The Strategic Determination of the Supply of Liquid Assets

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WEB APPENDIX

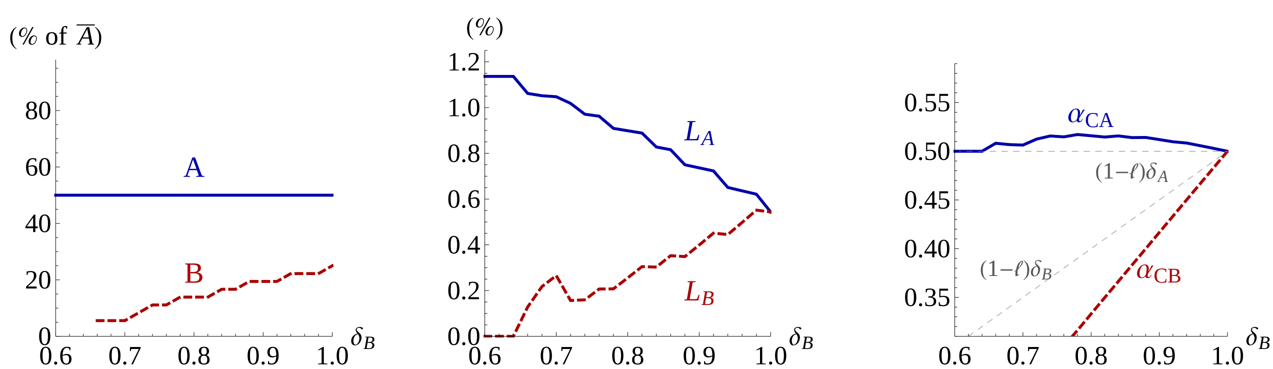
NOT FOR PUBLICATION

C Strategic Variants

In the paper, we analyze comparative statics of asset supplies and macro variables with respect to the parameters of OTC microstructure, assuming the issuers play a static Nash game with zero cost of issuing assets. Here, we analyze three variations of this model: (1) one issuer is non-strategic and the other best-responds; (2) one issuer is a Stackelberg leader and the other is a Stackelberg follower; (3) one issuer has a positive marginal cost of issuing assets.

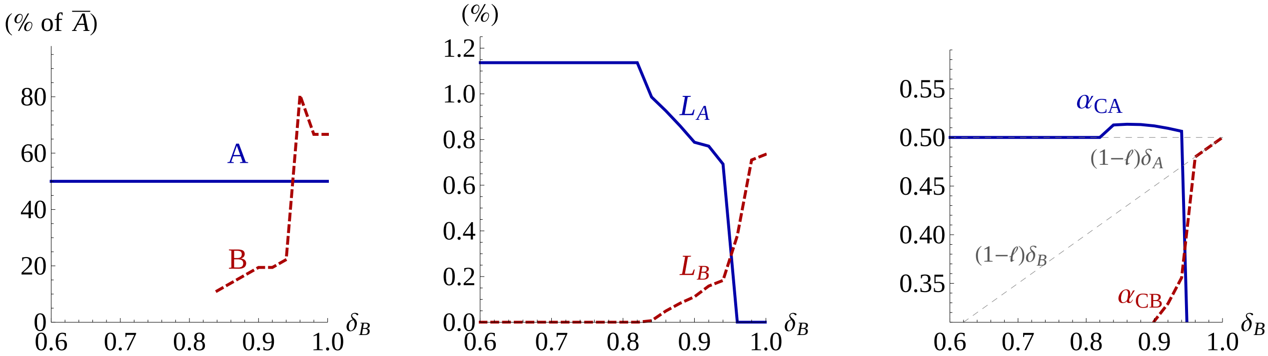
C.1 Analysis of the model with one non-strategic issuer

Suppose we hold A fixed, and let B best-respond. We call this case "semi-strategic" in distinction to the "fully strategic" case analyzed in Section 4.3. The incentives of issuer B are still identical to those characterized in Section 4.2; only, instead of looking at the full two-dimensional "playing field", we only need to be concerned with one vertical slice of it, corresponding to a fixed level of A.



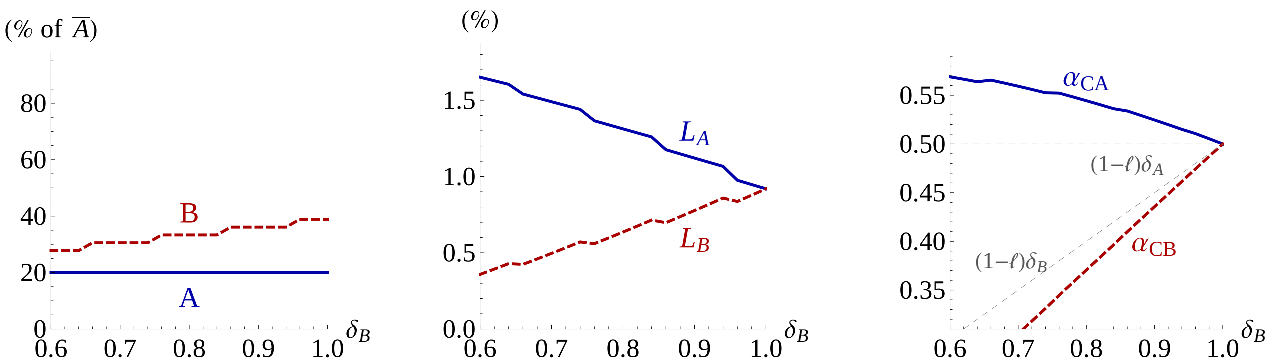
[a] Asset supplies [b] Net liquidity premia [c] OTC matching probabilities

Figure C.1: Comparative statics with respect to δ_B, of equilibria where B best responds to (large) $A = 0.5\bar{A}$, with CRS ($\rho = 0$).



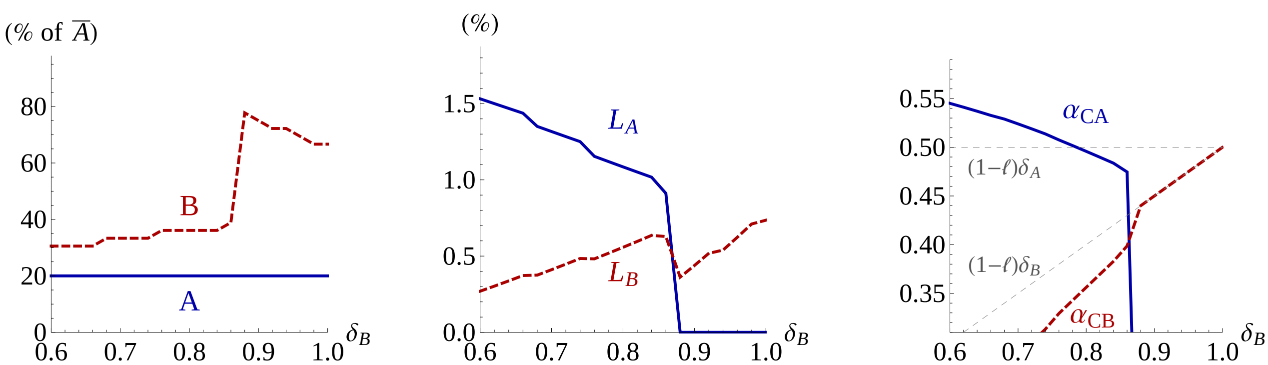
[a] Asset supplies [b] Net liquidity premia [c] OTC matching probabilities

Figure C.2: Comparative statics with respect to δ_B, of equilibria where B best responds to (large) $A = 0.5\bar{A}$, with IRS ($\rho = 0.1$).



[a] Asset supplies [b] Net liquidity premia [c] OTC matching probabilities

Figure C.3: Comparative statics with respect to δ_B, of equilibria where B best responds to (small) $A = 0.2\bar{A}$, with CRS ($\rho = 0$).



[a] Asset supplies [b] Net liquidity premia [c] OTC matching probabilities

Figure C.4: Comparative statics with respect to δ_B, of equilibria where B best responds to (small) $A = 0.2\bar{A}$, with IRS ($\rho = 0.1$).

We consider two values of A; "large" ($= 0.5\bar{A}$) and "small" ($= 0.2\bar{A}$). The large value is approximately what a monopolist would pick, or a Stackelberg leader. The small value is shown for comparison, as it makes it easier for B to outcompete A for secondary market liquidity. We maintain throughout that $\delta_A \geq \delta_B$. The interpretation is that A is a dominant issuer, such as the U.S. Treasury, who never has any exogenous *dis*advantages in secondary markets.

Figures C.1-C.4 show comparative statics of δ_B, as in the main text, for four numerical experiments: large A versus small A, CRS versus a moderate amount of IRS ($\rho = 0.1$). As before, lines are only shown in the graph where the result is determinate. For example, the quantity B is omitted in regions where $e_C = 1$, where B's premium is zero and she is indifferent to issuing any amount.

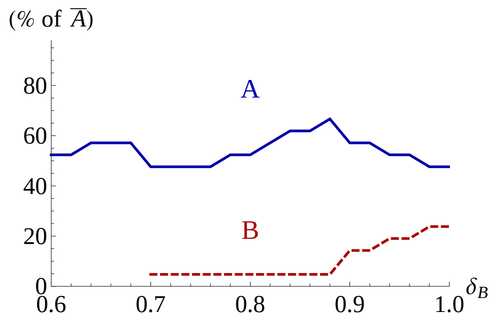
The results are straightforward: a smaller δ_B reduces the gains from entering the B-market. N-types are more sensitive to this, which compounds the effect from the C-types' point of view ($\alpha_{CB} < (1 - \ell)\delta_B$). As a result, B's liquidity premium is significantly reduced which in turn prompts B to issue less. With IRS, these effects are amplified – except for the fact that for δ_B large enough, issuer B takes advantage and issues so much that all secondary trade concentrates in the B-market. As we saw in the main text, this would not have been possible with two strategic players; A would have protected its secondary market by issuing more in turn.

Now, does this mean that AAA corporate bonds would become more liquid than Treasuries, if only their supply was large enough? We do not want to take the model so literally, as other considerations may come into play. For example, suppose that there is a positive (and perhaps upward sloping) cost of issuing safe debt – in that case, issuer B may not be capable of issuing (much) more than A. Or, suppose that there exists a fraction of investors who, for regulatory or tax reasons, prefer to hold A-assets – in that case, market A would not dry up completely (although B could still become a little bit more liquid than A). Finally, even if concentrating trade in the B-market was advantageous for the agents, it might still not happen because the A-corner is always an equilibrium. As we have said before, if traders have formed a habit of trading in the A-market in the past, it may take more than a theoretical benefit to get them to make the collective switch to B.

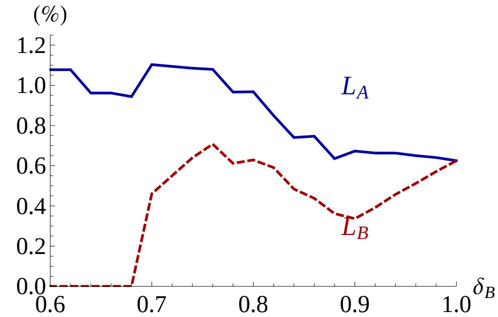
C.2 Analysis of the model with Stackelberg duopoly

Suppose we let A move first and issue a quantity of assets, then let B best-respond. This is the Stackelberg model of duopoly in contrast to the Cournot-Nash model analyzed in Section 4.3. The incentives of issuer B are still identical to those characterized in Section 4.2.

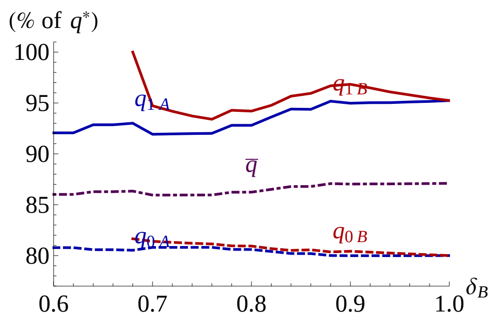
We repeat two of the three experiments from the main text: CRS in the matching function (as in Figure 7) and a small amount of IRS ($\rho = 0.02$, as in Figure 9). We maintain throughout that $\delta_A \geq \delta_B$, and look at comparative statics with respect to δ_B. They are shown in Figures C.5-C.6.



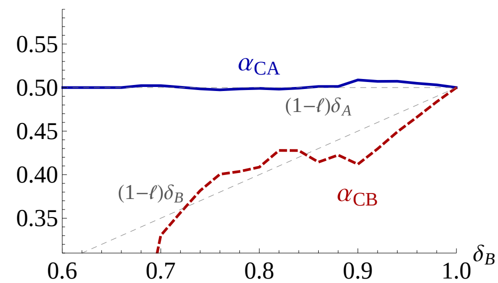
[a] Asset supplies



[b] Net liquidity premia

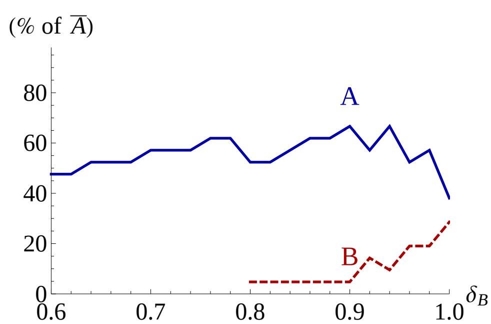


[c] Quantities traded in DM, and average

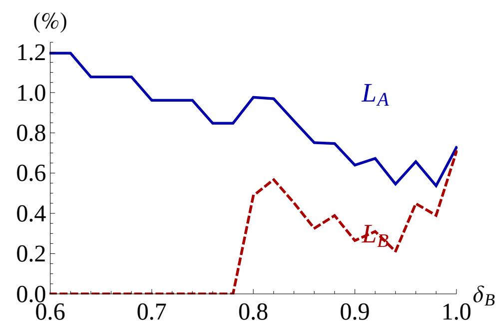


[d] OTC matching probabilities

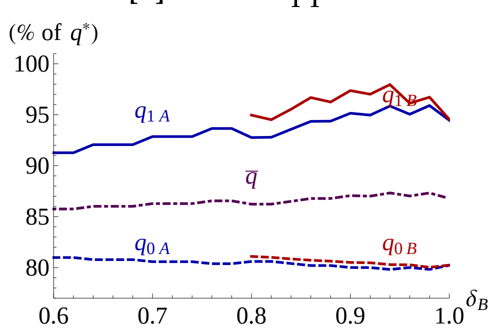
Figure C.5: Comparative statics with respect to δ_B, of Stackelberg equilibria where A moves first, with CRS ($\rho = 0$).



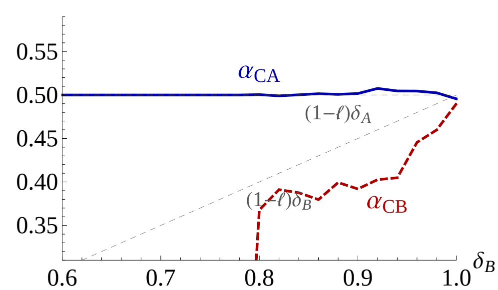
[a] Asset supplies



[b] Net liquidity premia



[c] Quantities traded in DM, and average



[d] OTC matching probabilities

Figure C.6: Comparative statics with respect to δ_B, of Stackelberg equilibria where A moves first, with a small amount of IRS ($\rho = 0.02$).

As before, lines are only shown in the graph where the result is determinate. For example, the quantity B is omitted in regions where $e_C = 1$, where B's premium is zero and she is indifferent to issuing any amount.

We can summarize the results as follows. When markets are identical ($\delta_A = \delta_B$), then A will issue approximately twice as much as B (which is the standard Stackelberg solution to a duopoly with linear demand). As δ_B decreases, A issues more and B issues less, but much less aggressively so than in the Nash case (Figures 7-9 in the main text). Why? The reason is that B now issues much less, so there is less incentive for A to ramp up their own issue and drive B out. Issuer A gets most of the market anyway by virtue of being the first mover, so a small amount of B-issue can be accommodated.

C.3 Analysis of the model where issuer B pays a cost to issue assets

Suppose that issuer B must pay a real marginal cost $\gamma_B \geq 0$ in order to create bonds. Specifically, this means that given B^- issued bonds in the previous period, their Bellman equation is:

$$W^B(B^-) = \max_{X,H,B} \left\{ X - H + \beta W^B(B) \right\}$$
$$\text{s.t. } X + \varphi B^- + \gamma_B B = H + \varphi p_B B,$$

which we can simplify to yield:

$$W^B(B^-) = -\varphi B^- + \max_{B} \left\{ \varphi p_B B - \gamma_B B + \beta W^B(B) \right\}.$$

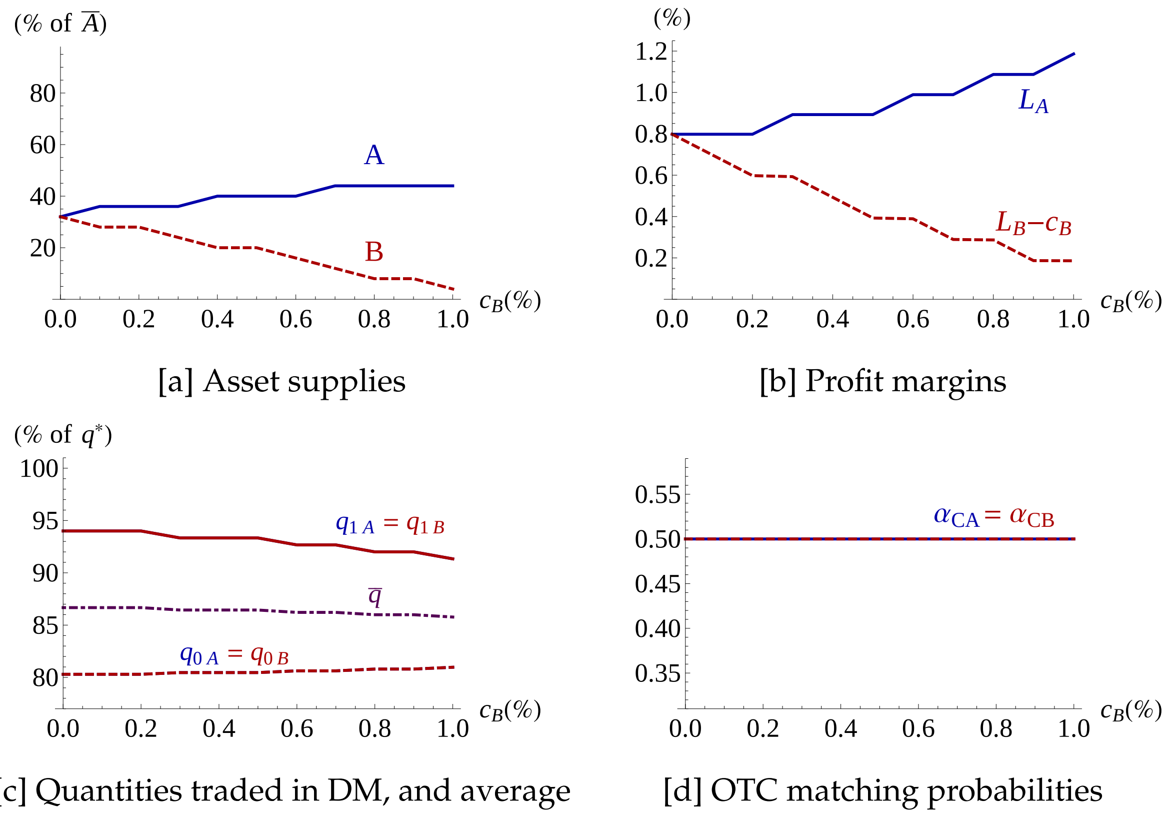
Just as before, the issuer's choice of B does not depend on their previous choices. We use this, plus the fact that in steady state $\varphi/\hat{\varphi} = (1 + \mu)$, to solve for issuer B's objective function:

$$J^B = \frac{\varphi}{1+i} \big(\ell \, \alpha_{CB} \theta \left[u'(q_{1B}) - 1 \right] - (1+i)\gamma_B \big) B. \tag{C.1}$$

Thus, the issuer's objective is equivalent to maximizing the product of their asset supply and the difference between the liquidity premium $L_B = \ell \, \alpha_{CB} \theta \left[u'(q_{1B}) - 1 \right]$, and the *effective marginal cost*, which we define to be $c_B \equiv (1 + i)\gamma_B$ for convenience:

$$\max_{B} \left(L_B - c_B \right) \cdot B.$$

For the sake of brevity, we restrict attention to the CRS case ($\rho = 0$) and the Nash equilibrium solution of the strategic game, and consider two market structures: balanced CRS where $\delta_A = \delta_B = 1$, and unbalanced CRS where the A-market has an exogenous matching advantage: $\delta_A = 1$ but $\delta_B = 0.9$. We then look at comparative statics with respect to the marginal issue



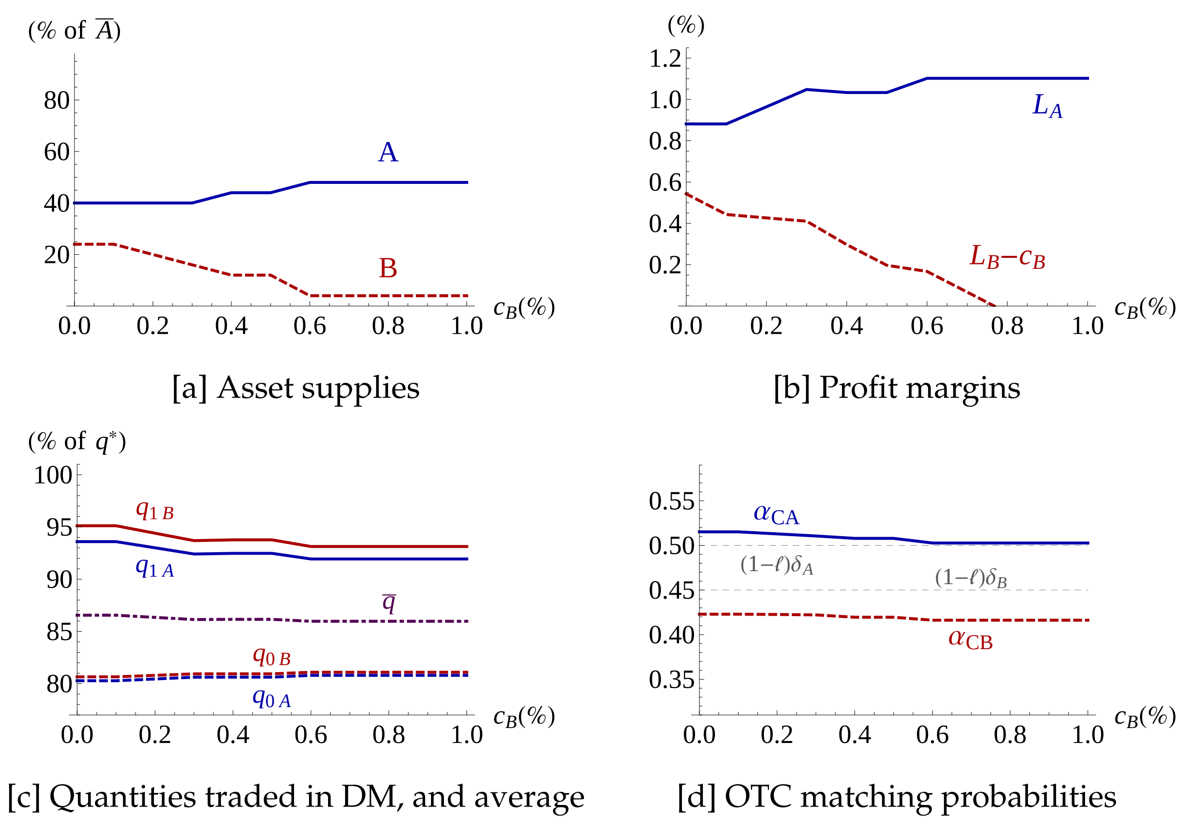
[a] Asset supplies

[b] Profit margins

[c] Quantities traded in DM, and average

[d] OTC matching probabilities

Figure C.7: Comparative statics with respect to c_B, with balanced CRS ($\rho = 0$, $\delta_A = \delta_B = 1$).



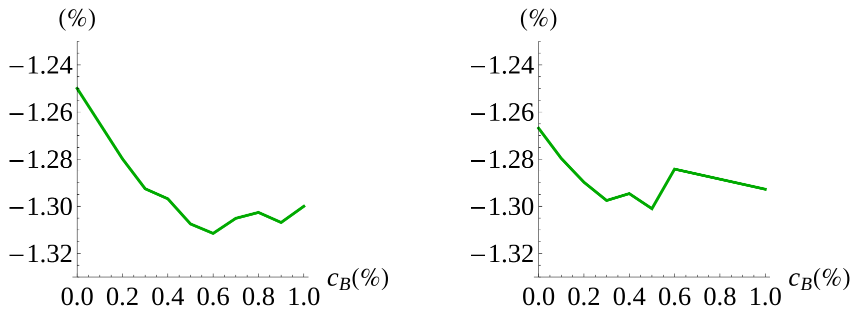
[a] Asset supplies

[b] Profit margins

[c] Quantities traded in DM, and average

[d] OTC matching probabilities

Figure C.8: Comparative statics w.r.t. c_B, with unbalanced CRS ($\rho = 0$, $\delta_A = 1$, $\delta_B = 0.9$).



[a] CRS and $\delta_A = \delta_B = 1$ [b] CRS, $\delta_A = 1$ and $\delta_B = 0.9$

Figure C.9: Welfare as a function of c_B, measured as equivalent CM consumption in percent deviations from the first-best, and including the cost of issuing asset B.

cost c_B. The results are shown in Figure C.7 (equilibrium outcomes for the balanced case), Figure C.8 (unbalanced case), and Figure C.9 (welfare in both cases).

The results for the balanced case are completely straightforward. No matter what B's cost of issuing is, as far as traders are concerned assets A and B still look like perfect substitutes. Thus, liquidity premia, matching probabilities, and real quantities are identical. The only thing that changes is that as B's cost increases, they issue a smaller amount of assets which allows issuer A to capture a larger share of the market.

As a consequence, social welfare is non-monotonic: as c_B increases above zero, B's issue sizes do not decrease fast enough, so the welfare impact is governed by the increasing issue costs. As issue costs become large enough, however, B is effectively driven out of the market so at that point, welfare increases slightly again. However, with B out of the market, issuer A acts as a monopolist, and we have already seen in the main text (Section 4.5) that for balanced CRS and for intermediate OTC bargaining power ($\theta = 0.5$), the Cournot duopoly outcome is better for social welfare than the monopoly outcome.

The results for the unbalanced case are also as one would expect. Even if B's cost of issuing is zero, the matching disadvantage in the B-market results in a lower issue size and liquidity premium for asset B. (The divergence in matching probabilities for C-types – asset sellers in the OTC markets – shown in Panel [d] is yet another clear illustration of the amplification effect discussed in the main text.) Consequently, the cost increase needed to drive B out of the market altogether is not as high.

D Analytical Solutions

D.1 Proof of Proposition 1 in the main text

Proof. (a) Assume $e_N = 0$. We claim that $e_C = 0$ is a best response by all C-types, i.e., $\tilde{S}_{CA} < \tilde{S}_{CB}$ when $e_N = 0$. First, notice that when $e_C = 0$, that is, when nobody is holding asset A, $q_{0A} = q_{1A} \, (\equiv \bar{q})$ and $\tilde{S}_{CA} = -i\bar{q} + \ell(u(\bar{q}) - \bar{q})$. If the matching probability in OTC$_B$ was zero as well, then $q_{0B} = q_{1B} \, (= \bar{q})$ and $\tilde{S}_{CB} = \tilde{S}_{CA}$. So it remains to show that $d\tilde{S}_{CB}/d\alpha_{CB} > 0$, keeping in mind that q_{0B}, q_{1B}, and the price of asset B can change as well. However, since this is the individual's choice problem, they choose their portfolio (q_0, q_1) without considering how it can affect aggregate variables.

We differentiate:

$$\frac{d\tilde{S}_{CB}}{d\alpha_{CB}} = \Big(- i + L_B[(1-\theta)u'(q_{0B}) + \theta] + \ell[u'(q_{0B}) - 1] - \ell\alpha_{CB}\theta[u'(q_{0B}) - 1] \Big) \frac{dq_{0B}}{d\alpha_{CB}} \quad \dots$$

$$+ \Big(- L_B[(1-\theta)u'(q_{1B}) + \theta] + \ell\alpha_{CB}\theta[u'(q_{1B}) - 1] \Big) \frac{dq_{1B}}{d\alpha_{CB}} \quad \dots$$

$$- \frac{dL_B}{d\alpha_{CB}} \cdot [(1-\theta)(u(q_{1B}) - u(q_{0B})) + \theta(q_{1B} - q_{0B})] + \ell\theta[u(q_{1B}) - u(q_{0B}) - q_{1B} + q_{0B}].$$

Now, if we substitute the right-hand side of the money demand equation for i, the coefficient on dq_{0B} vanishes. Similarly, if we substitute the definition of the liquidity premium for L_B, the coefficient on dq_{1B} vanishes, because the individual agent was already choosing (q_0, q_1) optimally. We are left with:

$$\frac{d\tilde{S}_{CB}}{d\alpha_{CB}} = -\ell\frac{\theta}{\omega(q_{1B})}[u'(q_{1B}) - 1][(1-\theta)(u(q_{1B}) - u(q_{0B})) + \theta(q_{1B} - q_{0B})] \quad \dots$$

$$+ \ell\theta[u(q_{1B}) - u(q_{0B}) - q_{1B} + q_{0B}]$$

$$= \ell\frac{\theta}{\omega(q_{1B})}\frac{1}{q_{1B} - q_{0B}}\left(\frac{u(q_{1B}) - u(q_{0B})}{q_{1B} - q_{0B}} - u'(q_{1B})\right).$$

And this term is positive since $q_{1B} > q_{0B}$ and u is strictly concave, which proves that $\tilde{S}_{CB} > \tilde{S}_{CA}$, and thus $e_C = 0$ is a best response to $e_N = 0$.

Then, with $e_C = 0$, we have $\alpha_{NA} = 0$ while $\alpha_{NB}S_{NB} > 0$ as long as asset B is in nonzero supply. Therefore, $G(0) < 0$ and $e_N = e_C = 0$ is an equilibrium.

(b) The proof is identical with the proof of (a), except that $e_N = e_C = 0$ is replaced with $e_N = e_C = 1$, and A and B are flipped.

(c) To find the limit as $e_N \to 0$, we (provisionally) guess that e_C/e_N is constant in the limit, which implies that α_{CA} and α_{NA} are constant as well. It does not imply that α_{CB} and α_{NB} are constant, since they depend on the ratio $(1 - e_C)/(1 - e_N)$; however, since both e_N and e_C are converging to zero, the ratio $(1-e_C)/(1-e_N)$ converges to 1. Thus, near the limit, α_{CB} and α_{NB} are approximately constant as well given our guess that e_C/e_N is constant in the limit.

Now, the question is what happens to the surpluses of C- and N-types that determine the entry choices, which in turn depend on the equilibrium trade quantities $(q_{0A}, q_{1A}, q_{0B}, q_{1B})$. First, look at the OTC bargaining solution (when C makes the offer): clearly, $e_C \downarrow$ implies $q_{1A} \uparrow$ and $q_{1B} \downarrow$ via the asset concentration/dilution effect. Also clearly, since e_C gets arbitrarily small, we must hit the point where $q_{1A} = q^*$ for a positive value of e_C. By our provisional result that the matching probabilities are constant, the money demand equation forces q_{0A} to stay constant, too; but then, since $q_{1A} = q^*$, the surplus $\tilde{S}_{CA}$ the C-type gets from specializing in asset A (Equation 13 in the paper) is constant, as well.

What about $\tilde{S}_{CB}$? When we totally differentiate the definition of $\tilde{S}_{CB}$, under the maintained assumption that α's are constant, we get:

$$d\tilde{S}_{CB} = \left[-i + \ell\left(1 - \alpha_{CB}\frac{\theta}{\omega(q_{1B})}\right)[u'(q_{0B}) - 1] + \ell\alpha_{CB}\frac{\theta}{\omega(q_{1B})}[u'(q_{1B}) - 1]\right] \cdot dq_{0B} \quad \dots$$

$$+ \ell\alpha_{CB}\frac{\theta}{\omega(q_{1B})^2}[(1-\theta)(u(q_{1B}) - u(q_{0B})) + \theta(q_{1B} - q_{0B})]\,[-u''(q_{1B})] \cdot dq_{1B}.$$

The money demand equation implies that the term multiplying dq_{0B} equals 0. Thus, $\tilde{S}_{CB}$ is an increasing function of q_{1B}. Earlier, we had shown that q_{1B} is increasing in e_C. Thus, $e_C \to 0$ implies that $\tilde{S}_{CB}$ decreases, and thus $\tilde{S}_{CA} > \tilde{S}_{CB}$ near $e_C = 0$, which falsifies the guess that e_C/e_N is constant.

Since the guess that e_C/e_N is constant would imply $\tilde{S}_{CA} > \tilde{S}_{CB}$ near $e_C = 0$, the C-type's entry choice equation must instead imply that e_C/e_N is increasing when $e_N \to 0$. This, in turn, implies $\alpha_{CA}\downarrow$, which restores the equation $\tilde{S}_{CA} = \tilde{S}_{CB}$. On the flipside, it also implies $\alpha_{NA}\downarrow$ (more entry of C-types into the A-market makes it easier for N-types to match there), while $q_{1B}\downarrow$ and $q_{1A} \to q^*$ is still the case. As a result, $\alpha_{NA}S_{NA} > \alpha_{NB}S_{NB}$ near $e_N \to 0$, which proves the first part of the statement: $\lim_{e_N \to 0+} G(e_N) > 0 > G(0)$.

The second part of the statement – $\lim_{e_N \to 1-} G(e_N) < 0 < G(1)$ – again has an identical proof, but with $e_N, e_C \to 0$ replaced with $e_N, e_C \to 1$, and A and B flipped.

The third part of the statement (a robust interior equilibrium exists) follows from continuity: in between $\lim_{e_N \to 0+} G(e_N)$ and $\lim_{e_N \to 1-} G(e_N)$, there must be at least one e_N which satisfies $G(e_N) = 0$, with G sloping down in a neighborhood of that point.

(d) The proof of part (c) above used CRS ($\rho = 0$) in that a constant ratio e_C/e_N implies that α_{CA} and α_{NA} are constant as well. This is no longer the case for IRS (any $\rho > 0$). Instead, now $e_N, e_C \to 0$ implies $\alpha_{CA}, \alpha_{NA} \to 0$. Thus, the G-function is continuous in neighborhoods of $e_N = 0$ and $e_N = 1$. (The discontinuities at the corners shown in part (c) arise because the matching probabilities are discontinuous at $e_C = e_N \in \{0,1\}$.) As a result, when $\rho > 0$, the proofs of parts (a) and (b) apply to these neighborhoods rather than just to the corners.

Again, like in the proof of (c), the third part of the statement (an interior equilibrium exists) follows from continuity: in between $G(0)$ and $G(1)$, there must be at least one e_N which satisfies $G(e_N) = 0$. But robustness is not assured, since now G may cross 0 from below; see Figure 3 in the main text for an illustration.

(e) We use the guess-and-verify method. Guess that $e_C = e_N = A/(A+B)$ and $q_{0A} = q_{0B}$ and $q_{1A} = q_{1B}$; it is easy to confirm that this satisfies all of the equilibrium equations, and that it is the unique solution when the sum $A + B$ is small enough.

(f) All of Sections D.2-D.3 below. □

D.2 Explicit differentiation around a scarce-assets interior equilibrium

Assume an interior equilibrium ($e_C \in (0,1)$) where asset supplies are small enough that both assets are scarce in OTC trade and thus valued for liquidity in CM trade ($q_{1A} < q^*$, $q_{1B} < q^*$).

OTC outcome when C makes the offer (Equations 10-11)

$$q_{1A} = \min\left\{q^*, q_{0A} + \frac{1}{\theta}\frac{A}{M}\frac{e_C q_{0A} + (1-e_C)q_{0B}}{e_C} - \frac{1-\theta}{\theta}[u(q_{1A}) - u(q_{0A})]\right\},$$
$$q_{1B} = \min\left\{q^*, q_{0B} + \frac{1}{\theta}\frac{B}{M}\frac{e_C q_{0A} + (1-e_C)q_{0B}}{1-e_C} - \frac{1-\theta}{\theta}[u(q_{1B}) - u(q_{0B})]\right\}.$$

Focus only on the scarce branch, and totally differentiate (holding M constant):

$$\omega(q_{1A}) \cdot dq_{1A} = \left(\omega(q_{0A}) + \frac{A}{M}\right) \cdot dq_{0A} + \frac{A}{M}\frac{1-e_C}{e_C} \cdot dq_{0B} - \frac{A}{M}\frac{q_{0B}}{e_C^2} \cdot de_C \quad \dots \qquad \text{(D.1)}$$
$$+ \frac{e_C q_{0A} + (1-e_C)q_{0B}}{e_C} \cdot dA/M,$$
$$\omega(q_{1B}) \cdot dq_{1B} = \left(\omega(q_{0B}) + \frac{B}{M}\right) \cdot dq_{0B} + \frac{B}{M}\frac{e_C}{1-e_C} \cdot dq_{0A} + \frac{B}{M}\frac{q_{0A}}{(1-e_C)^2} \cdot de_C \quad \dots \qquad \text{(D.2)}$$
$$+ \frac{e_C q_{0A} + (1-e_C)q_{0B}}{1-e_C} \cdot dB/M.$$

Money demand (Equation 9)

$$i = \ell\left(1 - \alpha_{Cj}\frac{\theta}{\omega(q_{1j})}\right)[u'(q_{0j}) - 1] + \ell\,\alpha_{Cj}\frac{\theta}{\omega(q_{1j})}[u'(q_{1j}) - 1], \qquad \text{for } j = \{A, B\}.$$

Totally differentiate:

$$di = \ell\left(1 - \frac{\alpha_{Cj}\theta}{\omega(q_{1j})}\right)u''(q_{0j}) \cdot dq_{0j} + \ell\alpha_{Cj}\theta\frac{\omega(q_{0j})}{\omega(q_{1j})^2}\, u''(q_{1j}) \cdot dq_{1j} \quad \ldots \tag{D.3}$$
$$+ \, \ell\frac{\theta}{\omega(q_{1B})}\,[u'(q_{1j}) - u'(q_{0j})] \cdot d\alpha_{Cj}.$$

Note for interpretation that *all three* of the right-hand-side coefficients are negative, because $u'' < 0$, and $q0 < q1$ implies $u'(q_0) > u'(q_1)$.

The liquidity premium (page 20)

$$L_j \equiv (1+i)p_j - 1 = \ell\,\alpha_{Cj}\frac{\theta}{\omega(q_{1j})}\,[u'(q_{1j}) - 1]\,, \qquad \text{for } \; j = \{A, B\}.$$

Totally differentiate:

$$dL_j = \ell\alpha_{Cj}\frac{\theta}{\omega(q_{1j})^2}\,u''(q_{1j}) \cdot dq_{1j} + \ell\frac{\theta}{\omega(q_{1j})}\,[u'(q_{1j}) - 1] \cdot d\alpha_{Cj}. \tag{D.4}$$

C's entry choice (Equations 12-13)

$$S_{Cj} = \theta[u(q_{1j}) - u(q_{0j}) - q_{1j} + q_{0j}], \qquad \text{for } \; j = \{A, B\},$$
$$\tilde{S}_{Cj} = -iq_{0j} - L_j[(1-\theta)(u(q_{1j}) - u(q_{0j})) + \theta(q_{1j} - q_{0j})] + \ell[u(q_{0j}) - q_{0j}] + \ell\,\alpha_{Cj}S_{Cj}.$$

Totally differentiate:

$$d\tilde{S}_{Cj} = -\,q_{0j} \cdot di - [(1-\theta)(u(q_{1j}) - u(q_{0j})) + \theta(q_{1j} - q_{0j})] \cdot dL_j + \ell S_{Cj} \cdot d\alpha_{Cj} \quad \ldots \tag{D.5}$$
$$+ \left(-\,i + L_j\,\omega(q_{0j}) + \ell(u'(q_{0j}) - 1) - \ell\alpha_{Cj}\theta\,[u'(q_{0j}) - 1]\right) \cdot dq_{0j} \quad \ldots$$
$$+ \left(-\,L_j\,\omega(q_{1j}) + \ell\alpha_{Cj}\theta\,[u'(q_{1j}) - 1]\right) \cdot dq_{1j}.$$

After substituting money demand for i and the definition of the liquidity premium for L_j, the terms on dq_{0j} and dq_{1j} vanish. This is because of an envelope argument: the C-type chooses their asset holdings and their eventual market participation jointly. Thus, Equation (D.5) becomes:

$$d\tilde{S}_{Cj} = -\,q_{0j} \cdot di - [(1-\theta)(u(q_{1j}) - u(q_{0j})) + \theta(q_{1j} - q_{0j})] \cdot dL_j + \ell S_{Cj} \cdot d\alpha_{Cj}.$$

We now complete the linearization of the C-choice:

$$
\begin{aligned}
&q_{0A} \cdot di + [(1-\theta)(u(q_{1A}) - u(q_{0A})) + \theta(q_{1A} - q_{0A})] \cdot dL_A - \ell S_{CA} \cdot d\alpha_{CA} \\
&\quad = q_{0B} \cdot di + [(1-\theta)(u(q_{1B}) - u(q_{0B})) + \theta(q_{1B} - q_{0B})] \cdot dL_B - \ell S_{CB} \cdot d\alpha_{CB}. \qquad \text{(D.6)}
\end{aligned}
$$

N's entry choice (Equations 15-16)

The differential analysis is much easier with an unscaled version of the G-function. Thus, for the purposes of this appendix, we define:

$$
\begin{aligned}
&G(e_N) \equiv \alpha_{NA} S_{NA} - \alpha_{NB} S_{NB}, \\
\text{where} \quad &S_{Nj} = (1-\theta)[u(q_{1j}) - u(q_{0j}) - q_{1j} + q_{0j}], \qquad \text{for } \ j = \{A, B\}.
\end{aligned}
$$

Totally differentiate:

$$
\begin{aligned}
&S_{NA} \cdot d\alpha_{NA} + \alpha_{NA} \cdot dS_{NA} - S_{NB} \cdot d\alpha_{NB} + \alpha_{NB} \cdot dS_{NB}, \\
\text{and} \qquad &dS_{Nj} = (1-\theta)[u'(q_{1j}) - 1] \cdot dq_{1j} - (1-\theta)[u'(q_{0j}) - 1] \cdot dq_{0j}.
\end{aligned}
$$

Thus the N-choice differential equation is:

$$
\begin{aligned}
dG = &S_{NA} \cdot d\alpha_{NA} + \alpha_{NA}(1-\theta)[u'(q_{1A}) - 1] \cdot dq_{1A} - \alpha_{NA}(1-\theta)[u'(q_{0A}) - 1] \cdot dq_{0A} \quad \ldots \qquad \text{(D.7)} \\
&- S_{NB} \cdot d\alpha_{NB} - \alpha_{NB}(1-\theta)[u'(q_{1B}) - 1] \cdot dq_{1B} + \alpha_{NB}(1-\theta)[u'(q_{0B}) - 1] \cdot dq_{0B}.
\end{aligned}
$$

Arrival probabilities (Equations 1-2)

Using the matching function from page 7 and the arrival probability definitions from Equations (1)-(2):

$$
\begin{aligned}
\alpha_{CA} &= \delta_A \, e_N (1-\ell) \; [e_N(1-\ell) + e_C \ell]^{\rho-1}, \\
\alpha_{CB} &= \delta_B \, (1-e_N)(1-\ell) \; [(1-e_N)(1-\ell) + (1-e_C)\ell]^{\rho-1}, \\
\alpha_{NA} &= \delta_A \, e_C \ell \; [e_N(1-\ell) + e_C \ell]^{\rho-1}, \\
\alpha_{NB} &= \delta_B \, (1-e_C)\ell \; [(1-e_N)(1-\ell) + (1-e_C)\ell]^{\rho-1}.
\end{aligned}
$$

Totally differentiate:

$$d\alpha_{CA} = \frac{\alpha_{CA}}{\delta_A} \cdot d\delta_A - (1-\rho)\ell(1-\ell)\,\delta_A\, e_N\,\left[e_N(1-\ell) + e_C\ell\right]^{\rho-2} \cdot de_C \quad \dots$$
$$+ \left[\frac{\alpha_{CA}}{e_N} - (1-\rho)(1-\ell)^2\,\delta_A\, e_N\,\left[e_N(1-\ell) + e_C\ell\right]^{\rho-2}\right] \cdot de_N, \tag{D.8}$$

$$d\alpha_{CB} = \frac{\alpha_{CB}}{\delta_B} \cdot d\delta_B + (1-\rho)\ell(1-\ell)\,\delta_A\,(1-e_N)\,\left[(1-e_N)(1-\ell) + (1-e_C)\ell\right]^{\rho-2} \cdot de_C \quad \dots$$
$$- \left[\frac{\alpha_{CB}}{1-e_N} - (1-\rho)(1-\ell)^2\,\delta_B\,(1-e_N)\,\left[(1-e_N)(1-\ell) + (1-e_C)\ell\right]^{\rho-2}\right] \cdot de_N, \tag{D.9}$$

$$d\alpha_{NA} = \frac{\alpha_{NA}}{\delta_A} \cdot d\delta_A + \left[\frac{\alpha_{NA}}{e_C} - (1-\rho)\ell^2\,\delta_A\, e_C\,\left[e_N(1-\ell) + e_C\ell\right]^{\rho-2}\right] \cdot de_C \quad \dots$$
$$- (1-\rho)\ell(1-\ell)\,\delta_A\, e_C\,\left[e_N(1-\ell) + e_C\ell\right]^{\rho-2} \cdot de_N, \tag{D.10}$$

$$d\alpha_{NB} = \frac{\alpha_{NB}}{\delta_B} \cdot d\delta_B - \left[\frac{\alpha_{NB}}{1-e_C} - (1-\rho)\ell^2\,\delta_B\,(1-e_C)\,\left[(1-e_N)(1-\ell) + (1-e_C)\ell\right]^{\rho-2}\right] \cdot de_C \quad \dots$$
$$+ (1-\rho)\ell(1-\ell)\,\delta_B\,(1-e_C)\,\left[(1-e_N)(1-\ell) + (1-e_C)\ell\right]^{\rho-2} \cdot de_N. \tag{D.11}$$

D.3 Special case: symmetric equilibrium

If $\delta_A = \delta_B = \delta$ and $A = B$, then a symmetric equilibrium exists where $e_C = e_N = 0.5$, thus $\alpha_{CA} = \alpha_{CB} = \delta(1-\ell)2^{-\rho}$ and $\alpha_{NA} = \alpha_{NB} = \delta\ell\, 2^{-\rho}$, thus $q_{0A} = q_{0B}\,(\equiv q_0)$ and $q_{1A} = q_{1B}\,(\equiv q_1)$. Around this symmetric equilibrium, the differential equations take the following forms:

Arrival probabilities (Equations D.8-D.11)

$$d\alpha_{CA} = (1-\ell)2^{-\rho} \cdot d\delta_A - (1-\rho)(1-\ell)\ell\,\delta\, 2^{1-\rho} \cdot de_C + (\rho + \ell - \rho\ell)(1-\ell)\delta\, 2^{1-\rho} \cdot de_N, \tag{D.12}$$
$$d\alpha_{CB} = (1-\ell)2^{-\rho} \cdot d\delta_B + (1-\rho)(1-\ell)\ell\,\delta\, 2^{1-\rho} \cdot de_C - (\rho + \ell - \rho\ell)(1-\ell)\delta\, 2^{1-\rho} \cdot de_N,$$
$$d\alpha_{NA} = \ell\, 2^{-\rho} \cdot d\delta_A + (1-\ell+\rho\ell)\ell\,\delta\, 2^{1-\rho} \cdot de_C - (1-\rho)(1-\ell)\ell\delta\, 2^{1-\rho} \cdot de_N, \tag{D.13}$$
$$d\alpha_{NB} = \ell\, 2^{-\rho} \cdot d\delta_B - (1-\ell+\rho\ell)\ell\,\delta\, 2^{1-\rho} \cdot de_C + (1-\rho)(1-\ell)\ell\delta\, 2^{1-\rho} \cdot de_N.$$

If, in particular, the δ's are constant, then the differential terms are *anti-symmetric*:

$$d\alpha_{CA} = -d\alpha_{CB} \qquad \text{and} \qquad d\alpha_{NA} = -d\alpha_{NB}. \tag{D.14}$$

If, beyond this, $\rho = 0$ (CRS matching), then the differential terms are *cross-anti-symmetric*:

$$\begin{aligned} d\alpha_{CA} = -d\alpha_{NA} = -d\alpha_{CB} = d\alpha_{NB} \\ = -2\ell(1-\ell)\delta \cdot de_{C} + 2\ell(1-\ell)\delta \cdot de_{N}. \end{aligned}$$

But with constant δ's and $\rho = 1$ (congestion-free matching), we get a different simplification:

$$d\alpha_{CA} = -d\alpha_{CB} = (1-\ell)\delta \cdot de_{N} \qquad \text{and} \qquad d\alpha_{NA} = -d\alpha_{NB} = \ell\,\delta \cdot de_{C}.$$

Post-trade quantities (Equations D.1-D.2)

$$\omega(q_1) \cdot dq_{1A} = \left(\omega(q_0) + \frac{A}{M}\right) \cdot dq_{0A} + \frac{A}{M} \cdot dq_{0B} - \frac{A}{M} 4q_0 \cdot de_{C} + 2q_0 \cdot dA/M,$$

$$\omega(q_1) \cdot dq_{1B} = \left(\omega(q_0) + \frac{A}{M}\right) \cdot dq_{0B} + \frac{A}{M} \cdot dq_{0A} + \frac{A}{M} 4q_0 \cdot de_{C} + 2q_0 \cdot dB/M.$$

If, in particular, the asset supplies are constant, then adding up gives:

$$\omega(q_1) \cdot (dq_{1A} + dq_{1B}) = \left(\omega(q_0) + \frac{2A}{M}\right) \cdot (dq_{0A} + dq_{0A}). \tag{D.15}$$

D.3.1 The symmetric equilibrium has an anti-symmetric derivative

Consider next the money demand equation (D.3). Its coefficients are the same at the symmetric equilibrium, so we can add up for $j = \{A, B\}$:

$$\begin{aligned} 2di = \ell\left(1 - \frac{\theta}{\omega(q_1)}\delta(1-\ell)2^{-\rho}\right)u''(q_0) \cdot (dq_{0A} + dq_{0B}) \quad \dots \\ + \ell\theta\frac{\omega(q_0)}{\omega(q_1)^2}\delta(1-\ell)2^{-\rho}\, u''(q_1) \cdot (dq_{1A} + dq_{1B}) + \ell\frac{\theta}{\omega(q_1)}\left[u'(q_1) - u'(q_0)\right] \cdot (d\alpha_{CA} + d\alpha_{CB}). \end{aligned}$$

Using (D.14), the last term vanishes. And if, in particular, i is constant, we obtain:

$$0 = \left(1 - \frac{\theta}{\omega(q_1)}\delta(1-\ell)2^{-\rho}\right)u''(q_0) \cdot (dq_{0A} + dq_{0B}) + \theta\frac{\omega(q_0)}{\omega(q_1)^2}\delta(1-\ell)2^{-\rho}\, u''(q_1) \cdot (dq_{1A} + dq_{1B})$$

$$\iff \qquad dq_{1A} + dq_{1B} = -\frac{1 - \frac{\theta}{\omega(q_1)}\delta(1-\ell)2^{-\rho}}{\theta\frac{\omega(q_0)}{\omega(q_1)^2}\delta(1-\ell)2^{-\rho}}\,\frac{u''(q_0)}{u''(q_1)} \cdot (dq_{0A} + dq_{0B}). \tag{D.16}$$

Combining (D.16) with (D.15), we have shown full anti-symmetry of the differential system:

$$dq_{0A} = -dq_{0B} \qquad \text{and} \qquad dq_{1A} = -dq_{1B}. \tag{D.17}$$

As a consequence, we can split the system into two "halves" and solve for them separately.

Formulation of the "halved" differential system

We maintain the assumptions that the δ's, asset supplies, and i are constant, and we are left with a linear differential system in seven variables: $(dq_{0A}, d_{1A}, d\alpha_{CA}, d\alpha_{NA}, de_{C}, de_{N}, dG)$. This system satisfies six equations, which is enough solve for our ultimate object of interest: the derivative of the G-function, dG/de_{N}. The first equation is OTC trade (equation above D.15) with anti-symmetry (D.17) used to substitute dq_{0B}:

$$\begin{aligned} 0 &= \omega(q_1) \cdot dq_{1A} - \omega(q_0) \cdot dq_{0A} + \frac{A}{M} 4q_0 \cdot de_{C} \\ &= \omega(q_1) \cdot dq_{1A} - \omega(q_0) \cdot dq_{0A} + 2\Big[(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)\Big] \cdot de_{C}. \end{aligned} \tag{D.18}$$

where we used the OTC solution (equation above D.1) evaluated at the symmetric equilibrium to substitute out the asset supply A/M.

Then, money demand (D.3) on the A-branch, multiplied by (-1) to make all coefficients positive and dividing out the extra ℓ:

$$\begin{aligned} 0 = &\left(1 - \frac{\theta}{\omega(q_1)}\delta(1-\ell)2^{-\rho}\right)[-u''(q_0)] \cdot dq_{0A} + \theta\frac{\omega(q_0)}{\omega(q_1)^2}\delta(1-\ell)2^{-\rho}\left[-u''(q_1)\right] \cdot dq_{1A} \quad \ldots \\ &+ \frac{\theta}{\omega(q_1)}\left[u'(q_0) - u'(q_1)\right] \cdot d\alpha_{CA}. \end{aligned} \tag{D.19}$$

Then, the C-entry choice (D.6). Using the anti-symmetry results we have $d\tilde{S}_{CA} = -d\tilde{S}_{CB}$ and $dL_A = -dL_B$, which yields:

$$0 = \Big[(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)\Big] \cdot dL_A - \ell S_{CA} \cdot d\alpha_{CA}.$$

Substitute the liquidity premium differential with (D.4) and divide out the extra $\ell\theta$:

$$\begin{aligned} 0 = &\Big[(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)\Big]\delta(1-\ell)2^{-\rho}\frac{1}{\omega(q_1)^2}u''(q_1) \cdot dq_{1A} \quad \ldots \\ &+ \Big[(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)\Big]\frac{1}{\omega(q_1)}\left[u'(q_1) - 1\right] \cdot d\alpha_{CA} - \frac{S_{CA}}{\theta} \cdot d\alpha_{CA}. \end{aligned}$$

Collect terms on $d\alpha_{CA}$, then multiply by $(-\omega(q_1))$:

$$0 = \Big[(1-\theta)(u(q_1)-u(q_0)) + \theta(q_1-q_0)\Big]\delta(1-\ell)2^{-\rho}\,\frac{1}{\omega(q_1)}\,[-u''(q_1)]\cdot dq_{1A} \quad \dots$$
$$+ (q_1-q_0)\left[\frac{u(q_1)-u(q_0)}{q_1-q_0} - u'(q_1)\right]\cdot d\alpha_{CA}. \tag{D.20}$$

Since u is concave, and $q_0 < q_1$, both coefficients are now positive.

The fourth equation is the N-entry choice (D.7). Thanks to anti-symmetry, we have:

$$\begin{aligned} dG &= 2S_N\cdot d\alpha_{NA} + 2\alpha_{NA}(1-\theta)[u'(q_1)-1]\cdot dq_{1A} - 2\alpha_{NA}(1-\theta)[u'(q_0)-1]\cdot dq_{0A} \\ &= 2S_N\cdot d\alpha_{NA} + \delta\ell\,2^{1-\rho}\,(1-\theta)[u'(q_1)-1]\cdot dq_{1A} - \delta\ell\,2^{1-\rho}\,(1-\theta)[u'(q_0)-1]\cdot dq_{0A}. \end{aligned} \tag{D.21}$$

where $S_N \equiv (1-\theta)[u(q_1)-u(q_0)-q_1+q_0]$.

Auxiliary terms

To simplify the presentation, define the auxiliary terms:

$$\Psi \equiv \frac{\omega(q_1)-\theta\delta(1-\ell)2^{-\rho}}{\theta\delta(1-\ell)2^{-\rho}}, \qquad \Omega_0 \equiv \frac{u'(q_0)-\frac{u(q_1)-u(q_0)}{q_1-q_0}}{-u''(q_0)}, \qquad \text{and} \qquad \Omega_1 \equiv \frac{\frac{u(q_1)-u(q_0)}{q_1-q_0}-u'(q_1)}{-u''(q_1)}.$$

Lemma D.1. *They satisfy the following properties:*

(a) $\dfrac{\omega(q_1)}{\omega(q_0)}\Psi > 1$ *if* $\theta\delta(1-\ell) < 2^{\rho-1}$ *and* i *is sufficiently low.*

(b) $\Omega_0 = \dfrac{1}{2}(q_1-q_0) + o\Big((q_1-q_0)^2\Big)$.

(c) $\Omega_1 = \dfrac{1}{2}(q_1-q_0) + o\Big((q_1-q_0)^2\Big)$.

(d) $\Omega_0, \Omega_1 > 0$ *for all* $q_1 > q_0$.

(e) $\Omega_0 \le \Omega_1$ *for all* $q_1 > q_0$ *if* $u''' \ge 0$.

Proof. (a) First notice that $\Psi \ge \frac{1-\theta\delta(1-\ell)2^{-\rho}}{\theta\delta(1-\ell)2^{-\rho}}$ and the fraction on the right-hand side is greater than 1 if $\theta\delta(1-\ell) < 2^{\rho-1}$. Since $\frac{\omega(q_1)}{\omega(q_0)} \le 1$ and $\frac{\omega(q_1)}{\omega(q_0)} \nearrow 1$ as $i \to 0$, there exist a sufficiently low i for which $\frac{\omega(q_1)}{\omega(q_0)}\Psi > 1$. (b) and (c) follow from applying L'Hospital's rule. (d) follows from strict concavity of u. (e) Expand all fractions and rewrite the inequality (keeping in mind that $u'' < 0$

by the assumption of strict concavity):

$$[-u''(q_1) - u''(q_0)]\,[u(q_1) - u(q_0)] - [-u''(q_0)]\,u'(q_1)(q_1 - q_0) - [-u''(q_1)]\,u'(q_0)(q_1 - q_0) \geq 0.$$

Define the left-hand side of the inequality to be Δ. Clearly, $\Delta\big|_{q_0=q_1} = 0$; now, let q_1 increase. After some tedious algebra, we find that the derivative is:

$$\frac{d\Delta}{dq_1} = u'''(q_1)\Big[u'(q_0)(q_1 - q_0) - u(q_1) + u(q_0)\Big] + [-u''(q_1)]\,\Big[u'(q_1) - u'(q_0) - u''(q_0)(q_1 - q_0)\Big].$$

The first term in square brackets is positive since u is strictly concave; the last term in square brackets is non-negative whenever u' is convex. This is the case whenever $u''' \geq 0$; thus, $u''' \geq 0$ implies $d\Delta/dq_1 \geq 0$ for all $q_1 > q_0$. Since Δ starts at zero when $q_1 = q_0$ and can never decrease when q_1 increases, this proves claim (e). □

D.3.2 Explicit solution of the "halved" differential system

Start with Equation (D.20). Use it to solve for $d\alpha_{CA}$:

$$\begin{aligned} d\alpha_{CA} &= \frac{(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)}{\omega(q_1)(q_1 - q_0)} \cdot \delta(1-\ell)2^{-\rho} \cdot \frac{u''(q_1)}{\frac{u(q_1)-u(q_0)}{q_1-q_0} - u'(q_1)} \cdot dq_{1A} \\ &= -\frac{(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)}{\omega(q_1)(q_1 - q_0)} \cdot \frac{\delta(1-\ell)2^{-\rho}}{\Omega_1} \cdot dq_{1A}. \end{aligned} \tag{D.22}$$

Substitute this result into Equation (D.19), and rearrange to solve for dq_{1A}:

$$dq_{1A} = \frac{\omega(q_1) - \theta\delta(1-\ell)2^{-\rho}}{\theta\delta(1-\ell)2^{-\rho}} \cdot \frac{u''(q_0)}{u''(q_1)} \cdot \frac{\frac{u(q_1)-u(q_0)}{q_1-q_0} - u'(q_1)}{u'(q_0) - \frac{u(q_1)-u(q_0)}{q_1-q_0}} \cdot dq_{0A} = \Psi\frac{\Omega_1}{\Omega_0} \cdot dq_{0A}. \tag{D.23}$$

Substitute (D.23) back into (D.22) to obtain:

$$d\alpha_{CA} = -\frac{(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)}{\omega(q_1)(q_1 - q_0)}\,\frac{\omega(q_1) - \theta\delta(1-\ell)2^{-\rho}}{\theta\,\Omega_0} \cdot dq_{0A}.$$

Now we can substitute (D.12) to eliminate $d\alpha_{CA}$:

$$\begin{aligned} &\theta\delta\,(1-\ell)\,2^{1-\rho}\Big[(\ell - \rho\ell)\,\cdot de_{C} - (\rho + \ell - \rho\ell)\cdot de_{N}\Big] \\ &\qquad = \frac{(1-\theta)(u(q_1) - u(q_0)) + \theta(q_1 - q_0)}{\omega(q_1)(q_1 - q_0)}\,\frac{\omega(q_1) - \theta\delta(1-\ell)2^{-\rho}}{\Omega_0} \cdot dq_{0A}. \end{aligned}$$

After some algebra and rearranging, we obtain:

$$\begin{aligned}dq_{0A} &= \frac{\omega(q_1)(q_1-q_0)}{(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)}\,\frac{\Omega_0}{\omega(q_1)-\theta\delta(1-\ell)2^{-\rho}}\,\theta\delta\,(1-\ell)\,2^{1-\rho} \quad \ldots \\ &\quad \times\Big[(\ell-\rho\ell)\,\cdot de_C-(\rho+\ell-\rho\ell)\cdot de_N\Big] \\ &= \frac{\omega(q_1)(q_1-q_0)}{(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)}\,\frac{2\Omega_0}{\Psi}\,\Big[(\ell-\rho\ell)\cdot de_C-(\rho+\ell-\rho\ell)\cdot de_N\Big]. \end{aligned} \tag{D.24}$$

Next, combine (D.23) with (D.18) to eliminate dq_{1A}:

$$\omega(q_0)\cdot dq_{0A}=\omega(q_1)\cdot\Psi\frac{\Omega_1}{\Omega_0}\cdot dq_{0A}+2\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]\cdot de_C.$$

Solve for dq_{0A}:

$$dq_{0A}=-\frac{\Omega_0}{\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)}\times 2\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]\cdot de_C. \tag{D.25}$$

Then substitute (D.25) into (D.24) to eliminate dq_{0A}:

$$\begin{aligned}&\Psi\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]^2\cdot de_C \\ &\qquad=\omega(q_1)(q_1-q_0)\,\Big[\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)\Big]\Big[-(\ell-\rho\ell)\cdot de_C+(\rho+\ell-\rho\ell)\cdot de_N\Big].\end{aligned}$$

After some more algebra, we obtain the effect of an (exogenous) change in market entry of N-types on the (endogenous) market choice of C-types:

$$\boxed{\frac{de_C}{de_N}=\frac{(\rho+\ell-\rho\ell)\,\omega(q_1)(q_1-q_0)\Big[\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)\Big]}{\Xi}} \tag{D.26}$$

where Ξ denotes:

$$\Xi\equiv(\ell-\rho\ell)\,\omega(q_1)(q_1-q_0)\Big[\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)\Big]+\Psi\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]^2.$$

In principle, this term could be negative, or even blow up if Ψ is small enough (i.e., $\theta\delta(1-\ell)2^{-\rho}$ and q_1 are big enough). But following the results in Lemma D.1, we assume that $\theta\delta(1-\ell)<2^{\rho-1}$, i is sufficiently low, and $u'''\geq 0$ (marginal utility u' is convex), so that it is positive and finite. If in addition we have CRS in the matching function ($\rho=0$), then the term is strictly in $(0,1)$. (But keep in mind that the term is evaluated at the symmetric equilibrium, i.e., at the entry mid-point of $e_C=e_N=1/2$. The slope is not necessarily $\in(0,1)$ for e_N near the corners.)

Now we substitute this back into (D.25) and (D.23), obtaining:

$$\frac{dq_{0A}}{de_N} = -\frac{2(\rho+\ell-\rho\ell)\,\omega(q_1)(q_1-q_0)\,\Omega_0\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]}{\Xi},$$

$$\frac{dq_{1A}}{de_N} = -\frac{2(\rho+\ell-\rho\ell)\,\omega(q_1)(q_1-q_0)\,\Psi\Omega_1\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]}{\Xi}.$$

Finally, we substitute these, together with (D.13) and (D.26), into (D.21) and collect terms:

$$\begin{aligned}
dG &= 2S_N\Big[(1-\ell+\rho\ell)\ell\delta 2^{1-\rho}\cdot de_C - (1-\rho)(1-\ell)\ell\delta 2^{1-\rho}\cdot de_N\Big] \quad \ldots \\
&\quad + \delta\ell 2^{1-\rho}\,(1-\theta)\Big[[u'(q_1)-1]\cdot dq_{1A} - [u'(q_0)-1]\cdot dq_{0A}\Big] \\
&= \delta\ell 2^{2-\rho}\,S_N\,\frac{1}{\Xi}\begin{bmatrix}\rho\,\omega(q_1)(q_1-q_0)\Big[\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)\Big] \\ -(1-\rho)(1-\ell)\Psi\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]^2\end{bmatrix}\cdot de_N \quad \ldots \\
&\quad - \delta\ell 2^{2-\rho}\,\frac{1}{\Xi}\begin{bmatrix}(1-\theta)(\rho+\ell-\rho\ell)\,\omega(q_1)(q_1-q_0) \quad \ldots \\ \times\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]\Big[[u'(q_1)-1]\Psi\Omega_1-[u'(q_0)-1]\Omega_0\Big]\end{bmatrix}\cdot de_N.
\end{aligned}$$

The last term is negative, maintaining the assumptions of Lemma D.1. This term reflects the intensive margin, the marginal surplus of trading in the A-market; as one would expect, more people entering this market drives down the marginal surplus due to the **asset dilution effect**. The preceding term, on the other hand, could be positive or negative; it reflects the balance of the **congestion effect** and the **thick market effect**. When $\rho = 0$ (constant returns to scale), there is no thick market effect, and the slope of the N-entry choice function is negative:

$$\left.\frac{dG}{de_N}\right|_{CRS} = \frac{4\delta\ell\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]}{\ell\,\omega(q_1)(q_1-q_0)\Big[\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)\Big]+\Psi\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]^2} \quad \ldots$$
$$\times\begin{bmatrix}-(1-\ell)\Psi\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]S_N \\ -(1-\theta)\ell\,\omega(q_1)(q_1-q_0)\Big[[u'(q_1)-1]\Psi\Omega_1-[u'(q_0)-1]\Omega_0\Big]\end{bmatrix}$$

When $\rho = 1$ (congestion-free matching), on the other hand, there is a thick market effect but no congestion effect. In this case, the slope of the N-entry choice function could be positive or negative, depending on which remaining effect dominates:

$$\left.\frac{dG}{de_N}\right|_{\rho=1} = \frac{2\delta\ell}{\Psi\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]} \quad \cdots$$
$$\times \left[\begin{array}{l} \omega(q_1)(q_1-q_0)\Big[\Psi\Omega_1\,\omega(q_1)-\Omega_0\,\omega(q_0)\Big]\dfrac{S_N}{\Big[(1-\theta)(u(q_1)-u(q_0))+\theta(q_1-q_0)\Big]} \\ -\,(1-\theta)\,\omega(q_1)(q_1-q_0)\Big[[u'(q_1)-1]\Psi\Omega_1-[u'(q_0)-1]\Omega_0\Big] \end{array}\right]$$

In general, this could go either way; however, as we can see, the strength of the thick market effect is governed by *average surplus* (average over all assets traded) available in each market, while the strength of the asset dilution effect is governed by *marginal surplus* (of the marginal asset traded). Thus, the latter effect is most likely to dominate when asset supplies are small. When asset supplies are nearly-plentiful, on the other hand, so that $q_1 \approx q^*$ and $u'(q_1) \approx 1$, the thick market effect becomes dominant, and the entry-choice function is upward sloping.