A Model of Endogenous Direct and Indirect Asset Liquidity*

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Abstract

Economists often say that certain types of assets, e.g., Treasury bonds, are very 'liquid'. Do they mean that these assets are likely to serve as media of exchange or collateral (a definition of liquidity often employed in monetary theory), or that they can be easily sold in a secondary market, if needed (a definition of liquidity closer to the one adopted in finance)? We develop a model where these two notions of asset liquidity coexist, and their relative importance is determined *endogenously* in general equilibrium: how likely agents are to visit a secondary market in order to sell assets for money depends on whether sellers of goods/services accept these assets as means of payment. But, also, the incentive of sellers to invest in a technology that allows them to recognize and accept assets as means of payment depends on the existence (and efficiency) of a secondary market where buyers could liquidate assets for cash. The interaction between these two channels offers new insights regarding the determination of asset prices and the ability of assets to facilitate transactions and improve welfare. We also use our model to study the efficacy of recent policy interventions in asset markets.

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1 Introduction

Asset liquidity recently has been front and center in the fields of monetary economics and finance. Interestingly, while these two strands of the literature agree that asset liquidity is essential for the study of a number of important topics (such as asset pricing, the implementation of monetary policy, and others), they employ *different* definitions of the term. In monetary theory, liquidity is typically an attribute of the asset itself, and it refers to how easily it can be used to purchase consumption. In finance, liquidity is typically an attribute of the (secondary) market where the asset trades, and it refers to the ease with which an individual can sell the asset, if needed.[1] In reality, both of these approaches are relevant. Sometimes agents use assets directly, either as media of exchange or collateral, to purchase goods and services from sellers, as is typically assumed in monetary theory. Other times, agents with a consumption need sell (or, as we often say, 'liquidate') assets in a secondary market, and then use the cash to purchase goods or services; this notion of liquidity is closer to the one adopted in finance.

This discussion raises a number of questions. What determines whether assets can be used as payment in transactions between buyers and sellers or the buyer/asset holder must liquidate them in a secondary asset market? Similarly, when economists say that certain assets, such as Treasury bonds, are "very liquid" do they mean that it is easy to use them as means of payment to purchase commodities or that it is easy to sell them in a secondary market? (An analogous question arises for assets that are considered illiquid, e.g., municipal bonds.) Last, but not least, are these details regarding the different liquidity aspects of assets important for the determination of asset prices and for their ability to facilitate transactions and improve welfare?

To answer these questions, one must employ a model that encompasses both of these notions of liquidity. Developing such model is the first main contribution of this paper. We build on the Lagos and Wright (2005) framework, where certain frictions, such as anonymity and imperfect commitment, impede trade in commodity markets and make a medium of exchange or collateral necessary. Fiat money helps bypass these frictions by serving as means of payment. Alongside money, a real asset can also potentially serve as a facilitator of trade. However, due to asymmetric information regarding the quality of the asset, only a fraction of sellers recognize and accept it in transactions. (All sellers accept money). We determine this fraction endogenously by allowing sellers to invest in information about the asset. As long as some sellers do not accept assets as payment, the buyers who are "matched" with them cannot use assets directly to buy goods. But even these buyers can benefit from the asset's 'liquidity', as they can visit a secondary market and sell their assets for money. Following Duffie et al. (2005), we assume that this market is

[1] This argument is also highlighted by Lagos (2008). For examples of papers in the first strand of the literature, see Lagos, Rocheteau, and Wright (2017) and the references therein; for examples of papers in the second strand of the literature, see Duffie, Gârleanu, and Pedersen (2005) and the references therein.

an over-the-counter (OTC) market, characterized by search and bargaining.[2]

Therefore, in our model, like in reality, sometimes assets compete *with money* as direct media of exchange, and some other times they must be liquidated *for money* in a secondary asset market, upon the arrival of a consumption opportunity. To fix ideas, we will refer to the former type of liquidity as *direct* asset liquidity and to the latter as *indirect*. Our paper not only provides a theory where both of these notions of liquidity coexist, but one where their relative importance is determined *endogenously* as a function of two fundamental parameters: i) The information/transaction cost that sellers must incur in order to recognize and accept assets in trade; and ii) The efficiency of matching in the secondary OTC asset market.

Building on Lester, Postlewaite, and Wright (2012) (LPW), we adopt a model where sellers of goods who are not informed about the asset refuse to accept it. An important contribution of LPW is to endogenize the sellers' decision to invest in information/technology to distinguish high and low quality versions of the asset. Thus, LPW is a model of *direct* liquidity: assets are liquid to the extent that sellers invest in the information that allows them to recognize and accept them as means of payment. Consequently, and in the authors' own words, "in any situation where buyers and sellers are asymmetrically informed about the values of assets, exchange is hindered". But this statement seems incomplete. If a seller turns down a buyer's, say, T-Bills because she does not recognize them, that does not mean that trade is ruled out: the buyer could still go to the secondary market for Treasuries—where, importantly, recognizability is not an issue—sell some bonds and return to the seller with her preferred method of payment, i.e., cash.

The present paper adds precisely this channel, i.e., it adds *indirect* asset liquidity. However, it is imperative to highlight that our model of direct *and* indirect liquidity is greater than the sum of its parts because, in general equilibrium, the degree of direct asset liquidity affects and is affected by the degree of indirect liquidity: how likely an agent is to "visit" the secondary market to liquidate assets, crucially depends on whether the seller of the goods/services she wishes to purchase will accept these assets as payment. And, vice versa, the incentive of a seller to invest in the technology that allows her to recognize assets is affected by the existence (and efficiency) of a secondary market where the buyer can liquidate assets for money. Our model studies the interaction between these two channels, and delivers a number of new insights.

We start with a model where the fraction of sellers who accept assets is exogenous. Agents make their portfolio choice between money and the real asset without knowing whether the seller they will meet accepts assets or not. Ex post, some agents match with sellers who accept the real asset, and some with sellers who do not. Once this idiosyncratic uncertainty has been resolved, an obvious motive for trade arises: the latter agents need money because that is all they can use as payment, and the former are happy to exchange some money for real assets

[2] This is arguably an empirically relevant choice. In the United States, Neklyudov and Sambalaibat (2015) report that the fraction of aggregate asset trade volume that took place in OTC markets was around 87% in 2010.

because, in their hands, these two objects are equally effective media of exchange. These are precisely the types of trades that the secondary OTC market allows to materialize.

In this environment agents are willing to pay a liquidity premium for the asset if its marginal unit: i) helps them acquire more goods by serving as payment (direct liquidity), and/or ii) helps them acquire additional money in the OTC, thus, relaxing a binding cash constraint (indirect liquidity). The asset price will include a liquidity premium as long as its supply is not too plentiful, and we provide a detailed characterization of the parameter space for which each type of liquidity is relevant. This has consequences for the effect of monetary policy on asset prices. If asset supply is low and inflation is intermediate, the asset can be valued both for its direct and indirect liquidity. In this region, a higher inflation not only increases the asset price (an established result in the literature) but does so with a higher elasticity. If inflation increases further, real money balances are depressed. Eventually, we enter a region where even though the agent's asset holdings are low, they are high enough to purchase all the real money balances available in the OTC market. That is to say, the indirect liquidity motive vanishes, and this lowers the elasticity of the asset price with respect to inflation.

The next step is to endogenously determine the fraction of sellers who accept assets in trade. To do so, we analyze the best response of a typical seller who believes that a fraction λ of (other) sellers accept assets. If λ is large, that seller has a lot to lose by not acquiring information: a high λ implies that agents can use their assets as means of payment often, which depresses money balances and hurts sellers who chose to not acquire information and only accept money. On the other hand, when λ is high, the agents who need to sell assets for money in the OTC market are few, and those who are willing to provide money (because they can use assets for exchange) are many: with market tightness in their favor, agents who seek to boost their money holdings are likely to succeed. In sum, a higher λ induces agents to carry less money *ex ante*, but it implies that agents who turn out to need money for trade are more likely to acquire it *ex post* in the OTC market. The first force encourages sellers to acquire information, thus, promoting coordination and corner equilibria. The second force discourages sellers from acquiring information and tends to generate *stable* interior equilibria, where some agents use assets directly as payment, and some others visit the OTC to liquidate assets for money. We find interior equilibria especially interesting, because they are arguably more empirically relevant.

With these opposing forces at work multiplicity of equilibrium can easily arise, and the analysis can get complex. To provide a sharper characterization of equilibrium we calibrate the model to U.S. data and study it numerically. (A detailed discussion of all possible equilibria is relegated to the accompanying Web Appendix.) For a reasonable parametrization of the model, we find that a unique and stable interior equilibrium arises. Around this equilibrium an increase in the information cost that sellers must incur to recognize assets leads to a lower equilibrium λ. Moreover, we show that an increase in the efficiency of matching in the OTC market reduces the measure of sellers who acquire information: a more efficient OTC market allows buyers to take advantage of the *indirect* liquidity properties of the asset, thus reducing sellers' incentives to invest in the information that makes assets *directly* liquid. Put differently, our model predicts a negative relationship between an asset's direct and indirect liquidity. In Section 5.3 we provide some empirical support for this theoretical finding.

Our hybrid model of direct and indirect asset liquidity offers important new insights regarding policy intervention. Policy makers in developed countries have recently taken various measures to improve asset liquidity. One such example is the so-called "eligibility policy" promoted by central banks to allow broader classes of assets to serve as collateral.[3] In our model, this intervention could be thought of as a reduction of the transaction costs that agents must incur when using assets as collateral and, hence, an improvement of the asset's *direct* liquidity. Another example is the recent Dodd-Frank Act, whose objective was to improve certain assets' market liquidity, or, to use the language introduced in this paper, to improve their *indirect* liquidity.[4] These actions indicate that policy makers favor an improvement in asset liquidity. However, until now we do not have a rigorous examination of how effective these various measures can be in improving welfare.

Our analysis shows that improving an asset's direct versus indirect liquidity has different effects on welfare (thus, details matter). A decrease in the cost agents must pay to use an asset as collateral increases the number of meetings where assets serve as direct means of payment and enhances welfare. In contrast, an increase in OTC market efficiency is shown to hurt welfare in the calibrated model. The intuition is as follows: a more efficient OTC market reduces the number of sellers who accept assets and crowds out the asset's direct liquidity. As a result, an excess number of agents try to sell assets in the OTC market and fail to find buyers, even though the efficiency of matching has improved. Thus, our model suggests that measures aimed to increase the direct liquidity of assets have better chances of improving the economy's welfare.

The present paper is related to a large and growing literature that has pointed out the importance of asset liquidity for the determination of asset prices. Examples of such papers include Geromichalos, Licari, and Suarez-Lledo (2007), Lagos and Rocheteau (2008), Lagos (2010), Nosal and Rocheteau (2013), Andolfatto and Martin (2013), Andolfatto, Berentsen, and Waller (2013), Venkateswaran and Wright (2014), Rocheteau and Wright (2013), Hu and Rocheteau (2015), Jung and Lee (2015), Geromichalos, Lee, Lee, and Oikawa (2016), and Jung and Pyun (2016), among many others. In these papers, assets are 'liquid' because they can facilitate transactions in fric-

[3] See for example this report by the Committee on the Global Financial System of the BIS: https://www.bis.org/publ/cgfs53.htm

[4] Consider for example the following quote from Lester, Shourideh, Venkateswaran, and Zetlin-Jones (2018): "OTC markets have undergone significant changes in recent years as a result of regulatory initiatives. One of the primary consequences of these changes has been a *reduction in trading frictions* (emphasis added). In the U.S., for example, the DoddFrank Wall Street Reform and Consumer Protection Act has called for the introduction of Swap Execution Facilities in the market for interest rate swaps.

tional decentralized markets, by serving directly as means of payment or collateral.

The closest paper to ours is Lester et al. (2012) (LPW), who extend the aforementioned literature by endogenizing the measure of sellers who accept assets. We add to this work by explicitly modeling a secondary market, where agents who cannot use assets directly as payment can sell these assets for money. Incorporating this 'indirect liquidity' channel amounts to much more than just adding an empirically relevant ingredient to the LPW framework, as the interaction between direct and indirect liquidity offers a number of new insights. Except from the differences (in asset prices and welfare) that have already been discussed, our novel indirect liquidity channel dramatically changes the properties of equilibrium. In LPW, a seller's profit is always increasing in the fraction of other sellers who accept assets, making corners the only stable equilibria. Here, that channel is also present, but now a seller who chooses to stay uninformed may be better off when more sellers acquire information, because she will meet an agent who is more likely to have boosted her money holdings in the OTC market. This new force tends to generate *stable interior* equilibria, where only a fraction of sellers choose to accept assets in trade.[5]

Our model is related to a number of recent papers that exploit the idea of indirect liquidity, i.e., the fact that assets can be sold in a secondary market upon the arrival of a liquidity need. Examples include Geromichalos and Herrenbrueck (2016), Mattesini and Nosal (2016), Berentsen, Huber, and Marchesiani (2014, 2016), Han (2015), Herrenbrueck and Geromichalos (2017), Herrenbrueck (2019), and Madison (2019). In these papers, agents with an exogenous consumption opportunity visit the secondary market to sell assets because sellers never accept them as means of payment. Hence, this literature imposes a *cash-in-advance constraint*, dictating that only money can be used as means of payment. In our model, agents visit a secondary market to liquidate assets only if the seller they matched with refuses to accept assets, and whether sellers accept assets or not is determined *endogenously*. Thus, the present paper provides deeper micro-foundations for indirect asset liquidity models in at least *two dimensions*. First, it endogenizes whether assets can be used directly as means of payment. Second, it endogenizes the measure of agents who choose to go to the OTC market in order to buy or sell assets.

Finally, our work is related to the literature initiated by the pioneering work of Duffie et al. (2005), which studies how bargaining and search frictions in OTC financial markets affect asset prices and trade. Examples of such papers include Vayanos and Weill (2006), Weill (2007), Lagos and Rocheteau (2009), Uslu (2016), Bethune, Sultanum, and Trachter (2016) and Chang and Zhang (2018). Our paper differs from these papers because it introduces an OTC market into a monetary model (where assets also have direct liquidity properties); hence, in our model agents

[5] We find interior equilibria interesting both from a theoretical standpoint and because they are arguably more empirically relevant. Our reading of LPW is that the authors also agree with this assessment. But since in that paper the interior equilibrium is unstable, the authors explore an extension of the model, where the information cost is different for each seller. For certain parametric specifications of the distribution of costs among sellers, LPW can generate a stable interior equilibrium. For more details, see Section 4.

visit the OTC to sell assets because they need money, while in these papers agents trade because they differ in their valuation for the asset. Lagos and Zhang (2015) also consider an environment where gains from trade arise due to differences in asset valuation, but that model is a monetary one (hence, closer to ours) since agents who wish to buy assets must pay with money.

The rest of the paper is organized as follows. In Section 2 we provide a description of the model. In Section 3 we analyze equilibrium treating the fraction of sellers who accept assets as payment as exogenous, and in Section 4 we endogenize this important variable. In Section 5 we calibrate the model to U.S. data and discuss equilibrium welfare. Proofs of the paper's statements (lemmas and propositions) can be found in appendix A. The accompanying Web Appendix contains some results (and their proofs) that are useful but not essential for understanding the main body of the paper.

2 The Model

Time is discrete and continues forever. Each period consists of three sub-periods, where different economic activities take place. In the first sub-period agents trade in an Over-The-Counter asset market, characterized by search and bargaining, as in Duffie et al. (2005). We dub this market the OTC. In the second sub-period agents trade in a decentralized commodity market, which we dub the DM. In the third sub-period agents trade in a centralized, competitive market, henceforth referred to as the CM. Before going to the details, we offer an intuitive description of the role played by each market. The CM is the typical settlement market of Lagos and Wright (2005), where agents settle their old portfolios and choose new ones. The DM is a decentralized market characterized by anonymity and imperfect commitment, where agents meet bilaterally and trade goods and services; this can include the retail market, the market for investment goods, etc. Crucially, the frictions in the DM make a medium of exchange (henceforth, MOE) or collateral necessary in transactions, and which assets can serve this role will be determined endogenously. Since some agents may only be able to use money as means of payment, the OTC is placed before the DM so that these agents can visit it and rebalance their portfolios, i.e., sell assets for money. One can think of this market as the secondary market for Treasuries, corporate bonds, municipal bonds, etc.

Agents live forever and discount future between periods at rate $\beta \in (0, 1)$. There are two types of agents, buyers and sellers, distinguished by their roles in the DM. Each type's measure is normalized to 1. Buyers consume in the DM and CM and supply labor in the CM; sellers produce in the DM and consume and supply labor in the CM. All agents can transform one unit of labor in the CM into one unit of the CM good, which is the *numeraire*. The preferences of buyers and sellers within a period are given by $\mathcal{U}(X, L, q) = X - L + u(q)$ and $\mathcal{V}(X, L, q) = X - L - q$, respectively, where X denotes consumption of CM goods, L is labor supply in the CM, and q

stands for the amount of DM good traded. We assume that u is twice continuously differentiable, with $u' > 0$, $u'(0) = \infty$, $u'(\infty) = 0$, and $u'' < 0$. Let q^* denote the first-best level of trade in the DM, i.e., $\{q^* \equiv q : u'(q^*) = 1\}$. All goods are perishable between periods.

There are two types of assets: fiat money and a one-period real asset. Buyers can purchase any amount of money and the asset at (real) prices φ and ψ in the CM, respectively.[6] The supply of money is controlled by a monetary authority, and follows the rule $M_{t+1} = (1 + \mu)M_t$, with $\mu > \beta - 1$. New money is introduced if $\mu > 0$, or withdrawn if $\mu < 0$, via lump-sum transfers to buyers in the CM. Money has no intrinsic value, but it possesses the standard properties that make it an acceptable MOE in the DM, most notably it is *recognizable* by everyone in the economy. The supply of the real asset is fixed and denoted by $A > 0$. Each unit of the asset purchased today delivers one unit of numeraire in next period's CM.

Moving on to the DM, the important question is which assets can serve as means of payment in that market. Following LPW, we assume that all sellers (recognize and) accept money but, due to asymmetric information about the quality of real assets, only a fraction of sellers accept them in trade. More precisely, an asset can obtain a high or a low value and, for simplicity, it is assumed that the low-value asset is completely worthless. As LPW point out, one can think of the low-quality asset as "a bad claim to a good tree" (i.e., a counterfeit) or "a good claim to a bad tree" (i.e., a lemon); for our analysis this distinction does not matter. Buyers can produce the worthless asset at zero cost, and, as a result, sellers optimally choose to never accept an asset they do not recognize.[7] We refer to sellers who accept only money as type 1 sellers and to those who accept both money and assets as type 2 sellers, and we let $\lambda \in [0, 1]$ denote the measure of type 2 sellers. As a starting point, we treat λ as an exogenous parameter, but eventually we endogenize this term (in Section 4), by assuming that sellers can acquire information that allows them to recognize the real asset. To keep things simple, we assume without loss of generality that all buyers meet a seller in the DM (and vice versa). Within each match the terms of trade are determined by Kalai bargaining, with $\eta \in (0, 1)$ denoting the buyer's bargaining power.

After buyers have made their portfolio decision in the CM, but before they visit the DM, they discover which type of seller they will match with in the DM. A fraction λ of buyers will meet with type 2 sellers, and the rest will meet with type 1 sellers. For convenience, we will call the former type 2 buyers and the latter type 1 buyers. Clearly, the type of seller with whom a buyer matches in the DM determines which assets she can use as MOE. Buyers who turn out to be of type 1 will not be able to use their asset to buy the DM good, but they can visit the OTC market to sell some assets for cash, and type 2 buyers will be happy to (buy assets and) provide that

[6] In this model, sellers will never choose to hold assets, as long as they are priced at a liquidity premium: a seller's identity is permanent, so why would she pay a liquidity premium if she knows that she will never have a liquidity need (in the DM)? As a result, the interesting portfolio choices are made by the buyers.

[7] As explained in LPW, this is technically convenient because it helps avoid bargaining under asymmetric information, which is significantly more complicated. Rocheteau (2011) and Li, Rocheteau, and Weill (2012) study liquidity-related questions in models where assets that are not recognized may be partially accepted in trade.

cash, because for them real assets, as well as money, are acceptable MOE.

It should now be clear that the OTC market allows a more efficient reallocation of liquidity or, alternatively, it allows money to end up in the hands of the agents who value it most (the type 1 buyers). The matching technology in the OTC market is described by the constant returns to scale function $f(x, y) = \alpha \frac{xy}{x+y}$, where x, y are the measures of asset buyers and sellers, respectively, and α is a parameter that measures the matching efficiency. Clearly, here $x = \lambda$ and $y = 1 - \lambda$. Thus, the total number of matches per period is $f(\lambda, 1 - \lambda) = \alpha\lambda(1 - \lambda)$. To keep the analysis as simple as possible, we assume that type 1 buyers make a take-it-or-leave-it (henceforth, TIOLI) offer to type 2 buyers.[8] An implicit assumption made here is that trade in the OTC market does not suffer from asymmetric information problems. We think of the OTC as a market ran by dealers/intermediaries who specialize in trading a particular type of assets. Thus, while a seller in the DM may turn down the buyer's bonds, say, because she does not recognize them, this will never be an issue in the OTC.

3 Exogenous Asset Acceptance λ

3.1 Value Functions

The value function of a buyer who enters the CM with money and asset holdings (m, a) is

$$W(m, a) = \max_{X, L, \hat{m}, \hat{a}} \{X - L + \mathbb{E}[\Omega(\hat{m}, \hat{a})]\} \tag{1}$$
$$\text{s.t.} \quad X + \varphi\hat{m} + \psi\hat{a} = L + \varphi(m + \mu M) + a,$$

where hats denote next period's choices, and $\mathbb{E}[\Omega(\hat{m}, \hat{a})]$ denotes the expected continuation value of a buyer who enters the OTC market with the portfolio $(\hat{m}, \hat{a})$. Substituting $X - L$ from the budget constraint allows us to rewrite this value function as

$$W(m, a) = \varphi m + a + \Lambda. \tag{2}$$

As is standard in models that build on Lagos and Wright (2005), W is linear.

Next, the expected value function of a buyer who enters the OTC market with the portfolio (m, a) is given by

$$\mathbb{E}[\Omega(m, a)] = (1 - \lambda)\Omega_1(m, a) + \lambda\Omega_2(m, a), \tag{3}$$

[8] This is the interesting case; it is agents who plan to sell the asset 'down the road' (in the secondary market) who are crucial for the determination of the issue price (in the CM). If type 2 buyers made a TIOLI offer, the asset would never carry a liquidity premium because of its ability to be sold for cash in the OTC market.

where Ω_i represents the value function in the OTC market for a buyer of type $i = \{1, 2\}$. Letting χ denote the units of asset that the type 1 buyer transfers to the type 2 buyer in the OTC, and p the (dollar) price per asset, we can write

$$\Omega_1(m, a) = \alpha\lambda V_1(m + p\chi, a - \chi) + (1 - \alpha\lambda)V_1(m, a), \tag{4}$$

$$\Omega_2(m, a) = \alpha(1 - \lambda)V_2(m - p\chi, a + \chi) + [1 - \alpha(1 - \lambda)]V_2(m, a), \tag{5}$$

where $V_i(m, a)$ denotes the value function of a type i buyer who enters the DM with portfolio (m, a). The interpretation of the OTC value functions is straightforward. If the buyer turns out to be of type 1 (equation (4)), she will try to sell assets for cash in the OTC market. If she is successful, with probability $\alpha\lambda$, she will sell χ units of the asset and acquire $p\chi$ units of money, where p, χ will be determined through bargaining in the OTC market. If she is unsuccessful, with probability $1 - \alpha\lambda$, she will simply continue into the DM with her original money holdings. A similar interpretation applies to equation (5).

Finally, consider the value function in the DM. We have

$$V_i(m, a) = u(q_i) + \beta W(m - d_i^m, a - d_i^a), \tag{6}$$

where q_i, d_i^m, and d_i^a denote the amount of DM good, money, and real asset, respectively, that change hands in a DM meeting between a seller and a buyer of type i. These terms of trade will be determined in Section 3.2.

3.2 Bargaining in the DM and OTC market

Following Kalai's proportional bargaining solution, we can write the bargaining problem in a type i DM meeting between a seller and a buyer with portfolio (m, a) as

$$\max_{q_i, d_i^m, d_i^a} \{u(q_i) + W(m - d_i^m, a - d_i^a) - W(m, a)\}$$
$$\text{s.t.} \quad u(q_i) + W(m - d_i^m, a - d_i^a) - W(m, a) = \tfrac{\eta}{1-\eta}\left[-q_i + W^S(d_i^m, d_i^a) - W^S(0, 0)\right],$$

and the feasibility constraints $d_i^m \leq m$ and $d_i^a \leq a$. Of course, we have $d_1^a = 0$, by assumption. The terms W^S denote the seller's CM value function, which are also linear in both arguments. As is standard, the proportional bargaining solution maximizes the buyer's surplus subject to the constraint that a fixed proportion, $(1 - \eta)/\eta$, of this surplus is equal to the surplus of the seller. Exploiting the linearity of W and W^S allows one to further simplify the problem to

$$\max_{q_i, d^m, d^a} \eta \{u(q_i) - q_i\} \tag{7}$$
$$\text{s.t.} \quad (1 - \eta)u(q_i) + \eta q_i = \varphi d_i^m + d_i^a,$$

and $d_i^m \leq m$ and $d_i^a \leq a$. The following lemma summarizes the bargaining solution.

Lemma 1. *Define $z(q) \equiv (1-\eta)u(q) + \eta q$ and $m^* \equiv \frac{z(q^*)}{\varphi}$. Then, the solution to the bargaining problem in a type 1 meeting is:*

$$d_1^m(m) = \begin{cases} m^*, & \text{if } m \geq m^* \\ m, & \text{if } m < m^* \end{cases} \tag{8}$$

$$q_1(m) = \begin{cases} q^*, & \text{if } m \geq m^* \\ z^{-1}(\varphi m), & \text{if } m < m^* \end{cases} \tag{9}$$

and $d_1^a = 0$. The solution to the bargaining problem in a type 2 meeting is:

$$(d_2^m(m,a), d_2^a(m,a)) = \begin{cases} (d_2^{m*}, d_2^{a*}), & \text{if } \varphi m + a \geq z(q^*) \\ (m,a), & \text{if } \varphi m + a < z(q^*) \end{cases} \tag{10}$$

$$q_2(m,a) = \begin{cases} q^*, & \text{if } \varphi m + a \geq z(q^*) \\ z^{-1}(\varphi m + a), & \text{if } \varphi m + a < z(q^*) \end{cases} \tag{11}$$

where (d_2^{m*}, d_2^{a*}) is the set of pairs (d_2^m, d_2^a) that satisfies $\varphi d_2^m + d_2^a = z(q^*)$.

Proof. See the appendix. □

The term $z(q)$ represents the value of real liquid balances that induces the seller to produce q units. The amount of DM good a buyer can afford depends on the liquid assets that she carries; in type 1 meetings, only money can be used as MOE, while in type 2 meetings both money and assets are accepted. The rest is straightforward. If the value of the buyer's liquid assets exceeds $z(q^*)$, she will purchase the first-best quantity q^* and spend an amount of assets equal to $z(q^*)$. On the other hand, if the value of the buyer's liquid assets is below $z(q^*)$, she will hand all of them to the seller, only to obtain an amount of DM good which is lower than q^*.

Next, consider a meeting between a type 1 buyer with portfolio (m,a) and a type 2 buyer with portfolio $(\tilde{m}, \tilde{a})$ in the OTC market, and recall that the former agent makes a TIOLI offer to the latter. The bargaining problem is given by:

$$\max_{p,\chi} \{V_1(m + p\chi, a - \chi) - V_1(m,a)\}$$
$$\text{s.t.} \quad V_2(\tilde{m} - p\chi, \tilde{a} + \chi) - V_2(\tilde{m}, \tilde{a}) = 0,$$

and the feasibility constraints $\chi \in [-\tilde{a}, a]$ and $p\chi \in [-\tilde{m}, m]$. Since p was defined as the dollar price of one unit of asset in the OTC, $p\chi$ is the total monetary boost that the type 1 buyer can obtain by selling assets. After replacing the V functions from equation (6) and some algebra, we can re-write the OTC bargaining problem as:

$$\max_{p,\chi} \{u(q_1(m+p\chi)) - u(q_1(m)) + \varphi\left[d_1^m(m) - d_1^m(m+p\chi)\right] + \left[\varphi p\chi - \chi\right]\}$$
$$\text{s.t.} \quad [u(q_2(\tilde{m}-p\chi,\tilde{a}+\chi)) - u(q_2(\tilde{m},\tilde{a}))] + [\rho(\tilde{m},\tilde{a}) - \rho(\tilde{m}-p\chi,\tilde{a}+\chi)] = \varphi p\chi - \chi, \tag{12}$$

$\chi \in [-\tilde{a}, a]$ and $p\chi \in [-\tilde{m}, m]$, where $\rho(m,a) \equiv \varphi d_2^m(m,a) + d_2^a(m,a)$. These mathematical expressions illustrate economic insights that have been already discussed: surplus in the OTC market is generated as money gets transferred into the hands of the type 1 buyer, who can only use this object as a MOE. In return, the type 1 buyer gives some real assets to the type 2 buyer, which is a great deal since, for the latter agent, the real asset (as well as money) is an acceptable MOE. Of course, under the TIOLI assumption, the net surplus that ends up with the type 2 buyer is zero, as indicated by equation (12). The following lemma summarizes the solution to the bargaining problem in the OTC market.

Lemma 2. *Consider a meeting in the OTC market between a type 1 and a type 2 buyer with portfolios* (m,a) *and* $(\tilde{m},\tilde{a})$*, respectively. Define the "cutoff" level of asset holdings*

$$\bar{a}(m,\tilde{m}) = \begin{cases} \varphi(m^* - m), & \text{if } m + \tilde{m} \geq m^* \\ \varphi\tilde{m}, & \text{if } m + \tilde{m} < m^* \end{cases}$$

Then, the solution to the bargaining problem in the OTC market is given by

$$\chi(m,\tilde{m},a) = \begin{cases} \bar{a}(m,\tilde{m}), & \text{if } a \geq \bar{a}(m,\tilde{m}) \\ a, & \text{if } a < \bar{a}(m,\tilde{m}) \end{cases} \tag{13}$$

$$p(m,\tilde{m},a) = \frac{1}{\varphi} \tag{14}$$

Proof. See the appendix. □

The "cutoff" level $\bar{a}$ captures the amount of assets that type 1 needs to sell in order to acquire the case-specific optimal monetary transfer. Why is that transfer 'case-specific'? Because it depends on the money holdings of the two parties: if $m + \tilde{m} \geq m^*$, the money of the two agents pulled together allows the type 1 buyer to purchase q^* in the DM. In this case, the optimal (real) money transfer is $\varphi(m^* - m)$, i.e., the type 1 wants to acquire the amount of money that she is missing in order to afford q^*. If, on the other hand, $m + \tilde{m} < m^*$, the type 1 buyer will not be able to purchase q^*, even if she acquired all of the type 2's money. In this case, the optimal (real) monetary transfer is $\varphi\tilde{m}$, i.e., the type 1 buyer should acquire all the type 2's money.

Having defined the case-specific optimal money transfer, the remaining question is "Can the type 1 buyer afford it?" The answer depends on whether her asset holdings, a, are enough to cover the cutoff levels $\bar{a}$ (which, clearly, are also case-specific). If yes, then the type 1 will give away exactly $\bar{a}$ units of assets and obtain the optimal amount of money. If not, she will give away all of her assets and obtain a less-than-optimal amount of money. Notice that the OTC asset price is always equal to $1/\varphi$, i.e., $p\varphi = 1$. This simply says that the type 2 agent cannot purchase assets at a discount due to the fact that she has no bargaining power.

3.3 Optimal Choices

As is standard in models that build on Lagos and Wright (2005), the representative buyer's portfolio decision does not depend on her trading history. This decision is described by maximizing the agent's objective function, call it J, which can be derived as follows. First, substitute (4) and (5) into (3) to obtain an expression for $\mathbb{E}\left[\Omega(m, a)\right]$. Then, substitute that expression into (1), exploiting (2) and (6), and focus only on the terms that contain the portfolio choices $(\hat{m}, \hat{a})$ inside the maximum operator (the rest do not affect the agent's optimal choice). This yields:

$$\begin{aligned} J(\hat{m}, \hat{a}) \equiv & -\varphi\hat{m} - \psi\hat{a} \\ & + \beta\big\{(1-\lambda)\big[\alpha\lambda\big(u(q_1(\hat{m}+\chi/\hat{\varphi})) + \hat{a} - \chi\big) + (1-\alpha\lambda)\big(u(q_1(\hat{m})) + \hat{a}\big)\big] \\ & + \lambda\big[\alpha(1-\lambda)\big(u(q_2(\hat{m}-\tilde{\chi}/\hat{\varphi}, \hat{a}+\tilde{\chi})) + \hat{\varphi}(\hat{m}-\tilde{\chi}/\hat{\varphi}) + \hat{a} + \tilde{\chi} - \rho(\hat{m}-\tilde{\chi}/\hat{\varphi}, \hat{a}+\tilde{\chi})\big) \\ & + [1-\alpha(1-\lambda)]\big(u(q_2(\hat{m}, \hat{a})) + \hat{\varphi}\hat{m} + \hat{a} - \rho(\hat{m}, \hat{a})\big)\big]\big\}. \end{aligned} \quad (15)$$

Naturally, the first line of J represents the cost of purchasing the portfolio $(\hat{m}, \hat{a})$, and the remaining lines represent the expected benefit from carrying that portfolio into the next period. For instance, consider the second line, which captures the benefit from being a type 1 buyer (an event that takes place with probability $1-\lambda$). That buyer may be able to trade in the OTC market (with probability $\alpha\lambda$), in which case she can sell χ units of assets and boost her money holdings by $\chi/\hat{\varphi}$. If the buyer does not match in the OTC (with probability $1-\alpha\lambda$), she will simply continue to the DM with her original money holdings. The third and fourth lines, which capture the benefit from being a type 2 buyer, admit a similar interpretation.[9] It turns out that there are four relevant regions:

Region 1: $\hat{m} + \tilde{m} \geq m^*$, $\hat{a} \geq \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} \geq z(q^*)$. There is enough money in the OTC match to allow the type 1 to purchase q^* in the DM. If the agent is of type 1, her asset holdings allow her to acquire the critical amount of money $m^* - \hat{m}$. If the buyer is of a type 2, her total liquid assets allow her to purchase q^* in the DM.

Region 2: $\hat{m} + \tilde{m} \lesseqqgtr m^*$, $\hat{a} < \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$. If the buyer is of type 1, her asset holdings are not enough to acquire the optimal amount of money from the type 2. If she is of type 2, her total

[9] By Lemma 2, the amount of assets that changes hands in the OTC depends on the money and asset holdings of type 1, and (only) on the money holdings of type 2. Consequently, the amount of assets traded in the OTC when the representative buyer is type 1, χ, will typically be different than the amount traded when the representative buyer is type 2, $\tilde{\chi}$. Thus, if the representative buyer holds the portfolio $(\hat{m}, \hat{a})$ and expects her typical trading partner (in the OTC) to hold the portfolio $(\tilde{m}, \tilde{a})$, then χ depends on the terms $(\hat{m}, \tilde{m}, \hat{a})$, but $\tilde{\chi}$ depends on the terms $(\tilde{m}, \hat{m}, \tilde{a})$.

liquid balances are not enough to purchase q^* in the DM. (Whether $\hat{m} + \tilde{m}$ exceeds m^* or not is irrelevant because the asset holdings are scarce anyway).

Region 3: $\hat{m} + \tilde{m} < m^*$, $\hat{a} \geq \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$. The total money in the OTC meeting is not enough to allow the type 1 buyer to purchase q^* in the DM. If the buyer is of type 1, her asset holdings are enough to acquire all the all the money of the type 2 buyer. If the buyer is of type 2, her total liquid balances are not enough to purchase q^* in the DM.

Region 4: $\hat{m} + \tilde{m} < m^*$, $\hat{a} \geq \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} \geq z(q^*)$. The total money in the OTC meeting is not enough to allow the type 1 buyer to purchase q^* in the DM. If the buyer is of type 1, her asset holdings are enough to acquire all the money of the type 2 buyer. If the buyer is of type 2, her total liquid balances are enough to purchase q^* in the DM.

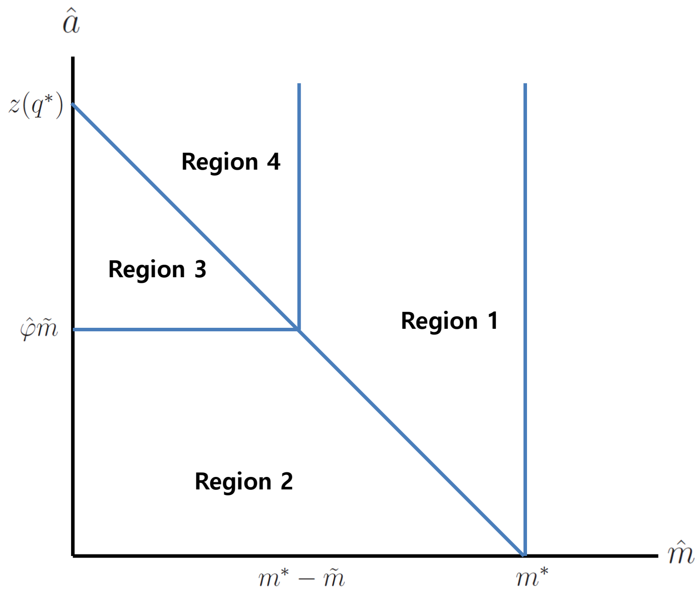

Figure 1: Regions of the individual optimization problem, in terms of money holdings.

Figure 1 illustrates the four regions. Why are they relevant? Because the region where the buyer finds herself in, is crucial for determining the benefit of the marginal unit of money/assets, which, of course, is crucial for determining the demand functions. Let us illustrate this through some examples. Suppose that given the price, $\hat{\varphi}$, and beliefs, $(\tilde{m}, \tilde{a})$, the representative buyer contemplates a portfolio choice that brings her in Region 1. Within that region, carrying an additional unit of the asset has no direct liquidity benefit (if I am a type 2 buyer I can already purchase q^*) or indirect liquidity benefit (if I am a type 1 buyer I can already acquire in the OTC the money I am missing in order to get to q^*). Hence, in that region, the buyer values an additional unit of asset only for its dividend, not for its liquidity. Does the buyer value an additional unit of money for its liquidity? Yes, because that unit helps her buy additional goods in the event of being a type 1 buyer who did not match in the OTC. As another example consider Region 2. Here, the marginal unit of real assets is valued both for its direct and indirect liquidity: direct, because $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$, so an additional unit of assets can help a type 2 buyer increase DM consumption, and indirect, because $\hat{a} < \bar{a}$, so an additional unit of assets can help a type 1 buyer acquire more money in the OTC.

We now provide a formal description of the representative buyer's optimal choice.

Lemma 3. *The function $J : \mathbb{R}^2 \to \mathbb{R}$ has the following properties:*

i. It is continuous and differentiable within all the regions.

ii. It is strictly concave in $\hat{m}$ and weakly concave in $\hat{a}$.

iii. It is weakly concave in its whole domain.

Let $J^i_j(\hat{m}, \hat{a})$, $j = 1, 2$, stands for the derivative of the objective function in Region $i = 1, 2, 3, 4$ with respect to the $j - th$ argument. Then, we have:

$$J^1_1(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\alpha\lambda + (1-\alpha\lambda)L(\hat{\varphi}\hat{m})\big] + \lambda\Big\}, \tag{16}$$

$$J^1_2(\hat{m}, \hat{a}) = J^4_2(\hat{m}, \hat{a}) = -\psi + \beta, \tag{17}$$

$$J^2_1(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\alpha\lambda L(\hat{\varphi}\hat{m} + \hat{a}) + (1-\alpha\lambda)L(\hat{\varphi}\hat{m})\big] + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{18}$$

$$J^2_2(\hat{m}, \hat{a}) = -\psi + \beta\Big\{(1-\lambda)\big[\alpha\lambda L(\hat{\varphi}\hat{m} + \hat{a}) + (1-\alpha\lambda)\big] + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{19}$$

$$J^3_1(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\alpha\lambda L(\hat{\varphi}(\hat{m} + \tilde{m})) + (1-\alpha\lambda)L(\hat{\varphi}\hat{m})\big] + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{20}$$

$$J^3_2(\hat{m}, \hat{a}) = -\psi + \beta\Big\{(1-\lambda) + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{21}$$

$$J^4_1(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\alpha\lambda L(\hat{\varphi}(\hat{m} + \tilde{m})) + (1-\alpha\lambda)L(\hat{\varphi}\hat{m})\big] + \lambda\Big\}, \tag{22}$$

where we have defined $L(\cdot) \equiv (h \circ z^{-1})(\cdot)$, with $h(q_i) \equiv \frac{u'(q_i)}{z'(q_i)}$. The liquidity premium function L satisfies $L(\cdot) \geq 1$ and $L'(\cdot) < 0$.

Proof. See the appendix. □

Lemma 4. *Taking prices, $(\varphi, \hat{\varphi}, \psi)$, and beliefs, $(\tilde{m}, \tilde{a})$, as given, the optimal portfolio choice of the representative buyer, $(\hat{m}, \hat{a})$, can be characterized as follows:*

i. It satisfies $J^i_j(\hat{m}, \hat{a}) = 0$, for all i, j.

ii. If $\varphi > \beta\hat{\varphi}$ and $\psi = \beta$, there exists a unique $\hat{m}$, whereas any $\hat{a}$ is optimal as long as $(\hat{m}, \hat{a})$ is in Regions 1, 4, or on the boundary between them.

iii. If $\varphi > \beta\hat{\varphi}$ and $\psi > \beta$, there exists a unique optimal portfolio choice $(\hat{m}, \hat{a})$, which lies in Regions 2, 3, or on the boundary between them.

Proof. See the appendix. □

Naturally, the optimal portfolio choice of the buyer amounts to equating the marginal cost of each asset (φ for money and ψ for the real asset) to its marginal benefit, which depends on the relevant region. If $\psi = \beta$, the net cost of carrying assets across periods is zero, thus, optimality dictates that the buyer bring an amount of assets high enough to exploit *all* possible liquidity properties (direct and indirect), and this can only happen in Regions 1 and 4. The buyer is only willing to buy the asset at a price higher than the fundamental value, i.e., $\psi > \beta$, if the marginal unit is still helpful for liquidity purposes, which is true only in Regions 2 and 3.

Of course, the asset's direct and indirect liquidity properties affect not only its own demand (and price), but also the demand for money. While interesting, examining the money demand is not of first-order importance for the analysis, so we relegate it to Section A of the Web Appendix.

3.4 Equilibrium

With the optimal behavior of the representative buyer laid out, it is now straightforward to characterize equilibrium, and we focus on symmetric, steady state equilibria, where $\varphi M = \hat{\varphi}\hat{M}$, implying that $\varphi/\hat{\varphi} = 1 + \mu$.

Definition 1. *A steady state equilibrium is a list $\{\psi, \chi, \mathrm{w}_1, \mathrm{w}_2, q_1^n, q_1^m, q_2^n, q_2^m\}$. The terms ψ, χ have already been defined. The remaining equilibrium objects are as follows: $\mathrm{w}_1 = \varphi M$ and $\mathrm{w}_2 = \varphi M + A$, represent the real liquid balances in a type 1 and a type 2 DM meeting, respectively; $q_1^n(q_2^n)$ stands for the amount of DM good traded in a type 1 (type 2) DM meeting, when the buyer was **not matched** in the preceding OTC market; $q_1^m(q_2^m)$ is the analogous expression for the case in which the buyer was **matched** in the OTC. The equilibrium objects are such that:*

i. *Given prices, the representative buyer's portfolio choice maximizes her objective function, i.e., it satisfies Lemma 4.*

ii. *The equilibrium quantity q_1^n is given by $q_1^n = z^{-1}(\mathrm{w}_1)$. The quantities q_1^m, q_2^n, and q_2^m can be obtained as follows:*

$$q_1^m = \begin{cases} q^*, & \text{in Region 1} \\ z^{-1}(\mathrm{w}_2), & \text{in Region 2} \\ z^{-1}(2\mathrm{w}_1), & \text{in Region 3 and 4} \end{cases}$$

$$q_2^n = q_2^m = q_2 = \begin{cases} q^*, & \text{in Region 1 and 4} \\ z^{-1}(\mathrm{w}_2), & \text{in Region 2 and 3} \end{cases}$$

iii. *The amount of assets traded in the OTC, χ, satisfies (13).*

iv. Each market clears and expectations are rational: $\hat{m} = \tilde{m} = (1+\mu)M$, *and* $\hat{a} = \tilde{a} = A$.

The amount of DM good traded depends on the type of meeting and, in the case of a type 1 meeting, on whether the buyer was matched in the preceding OTC market. If the type 1 buyer did not match, the amount of q she can purchase depends only on her own real balances, i.e., $q_1^n = z^{-1}(\mathrm{w}_1)$. If she did match, her post-OTC trade money balances depend on the specific region. In Region 1, both money and assets are plentiful, hence, the type 1 buyer will obtain q^*. In Regions 3, 4, assets are plentiful, but money is not. Hence, the type 1 buyer will acquire all the money of the type 2 (by symmetry, this implies that she will enter the DM with real balances equal to $2\mathrm{w}_1$), and purchase $q_1^m = z^{-1}(2\mathrm{w}_1) < q^*$. In Region 2, the type 1 buyer's assets do not allow her to purchase the optimal amount of money from type 2. Thus, the amount of DM good purchased by type 1 also depends on A, specifically, $q_1^m = z^{-1}(\mathrm{w}_2) < q^*$, where $\mathrm{w}_2 = \mathrm{w}_1 + A$. Notice that the amount of DM good purchased by a type 2 buyer, q_2, is irrelevant of whether she matched in the OTC, because that type has no bargaining power in the OTC market.

Lemma 5. *A steady state equilibrium exists and is unique.*

Proof. See the appendix. □

3.5 Characterization of Equilibrium

Having established equilibrium existence and uniqueness, we proceed to the characterization of equilibrium asset prices for an exogenous probability of asset acceptance, λ. To that end, it is useful to understand how the various equilibrium regions look on aggregate (Figure 1 illustrated these regions from the perspective of a representative agent). Figure 2 does precisely that, i.e., it illustrates the four equilibrium regions, not as functions of individual choices $\hat{a}, \hat{m}$, but as functions of the exogenous asset supply A (which, in equilibrium, equals $\hat{a}$), and the policy parameter μ (which, in equilibrium, is the main driver of $\hat{m}$).

While the details of the derivation of Figure 2 are relegated to Section B of the Web Appendix, the intuition is straightforward. In Region 1 type 2 and matched type 1 buyers are able to attain q^*. Naturally, this happens when A is relatively high and μ, the steady state inflation rate, is relatively low. Now, suppose that the asset supply is relatively high, say $A = A^h$ in the figure, and consider an increase in μ keeping A constant. As μ increases, equilibrium real balances w_1 decrease, and soon the matched type 1 buyer will not be able to acquire the money that would allow her to purchase q^* (although a type 2 buyer can still afford q^*). In a sense, the type 1's asset is plentiful, but the aggregate amount of money in the OTC is not. This is precisely what is going on in Region 4. If μ kept increasing, then the real balances w_1 would decrease so much that, eventually, even the type 2 buyer would not afford q^*. In other words, we would now be in Region 3.

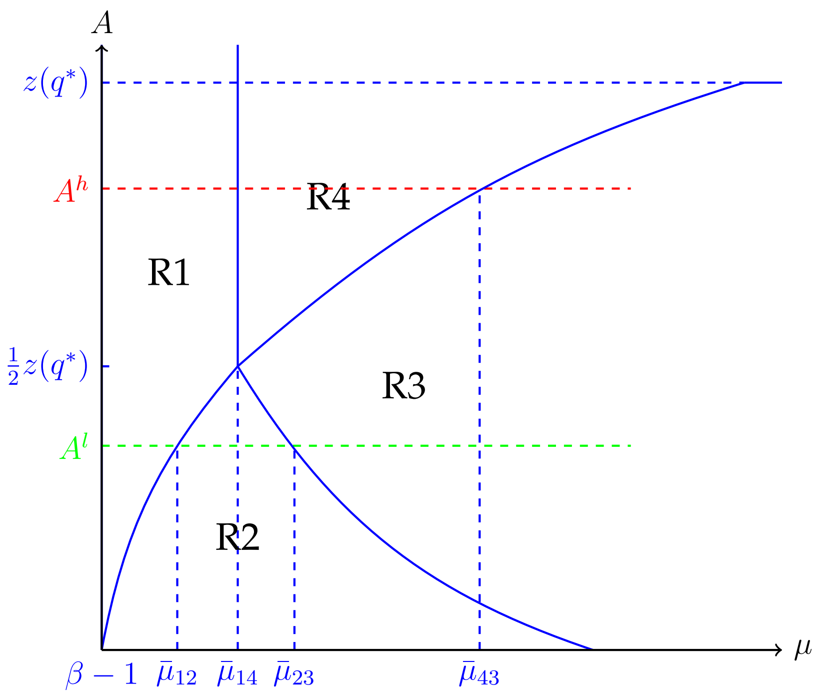

Figure 2: Aggregate regions of equilibrium, in terms of money growth and asset supply.

What if the asset supply was relatively low, say $A = A^l$, as in the figure? First notice that even for such low asset supply, we can still be in Region 1, but this would require an extremely low μ. As μ increases, the equilibrium real balances w_1 decrease. With A so low, and with inflation on the rise, but still in intermediate levels, we are in a region where type 2 buyers cannot afford q^*, and matched type 1 buyers do not have enough assets to acquire the amount of money they would wish in the OTC. As μ increases further, and with A fixed, an interesting development takes place: assets are still not enough to allow type 2 buyers to purchase q^*, but they are enough to allow type 1 buyers to acquire all the real money balances of type 2 buyers in the OTC, because these balances are now very little; in other words, we are now in Region 3.

The following proposition formalizes the results concerning equilibrium asset prices.

Proposition 1. *Let $\bar{\mu}_{ij}$, $i, j \in \{1, 2, 3, 4\}$, denote the values of μ that determine the boundary points between Regions i and j, for any given asset supply A; these boundaries are defined in Lemma WA.1 in the Web Appendix. Then, equilibrium asset prices are as follows:*

Case 1: If $A \geq z(q^*)$, then, for any $\mu > \beta - 1$, we have $\psi = \beta$;

Case 2: If $A \in (z(q^*)/2, z(q^*))$, then,

i. For all $\mu \in (\beta - 1, \bar{\mu}_{43}]$, we have $\psi = \beta$;

ii. For all $\mu > \bar{\mu}_{43}$, the CM asset price exceeds the fundamental value, and it is a strictly increasing function of μ, i.e., $\psi = \psi(\mu) > \beta$, and $\psi'(\mu) > 0$;

Case 3: If $A < z(q^*)/2$, then

i. For all $\mu \in (\beta - 1, \bar{\mu}_{12}]$, we have $\psi = \beta$;

ii. For all $\mu > \bar{\mu}_{12}$, the CM asset price exceeds the fundamental value, and it is a strictly increasing function of μ, i.e., $\psi = \psi(\mu) > \beta$, and $\psi'(\mu) > 0$.

Proof. See the appendix. □

The key observation is that agents are willing to pay liquidity premia, only if the marginal unit of the asset serves a (direct or indirect) liquidity purpose. Thus, for any $A \in (z(q^*)/2, z(q^*))$, there exists $\bar{\mu}_{43}$ such that $\mu \le \bar{\mu}_{43}$ will bring us in Region 1 or 4. Within these regions all the liquidity properties of the asset have been exploited and $\psi = \beta$. The same is true if $A < z(q^*)/2$ and $\mu \in (\beta - 1, \bar{\mu}_{12}]$, since these parameter values place equilibrium in Region 1. If $A < z(q^*)/2$ and $\mu \in (\bar{\mu}_{12}, \bar{\mu}_{23})$, equilibrium lies in Region 2, where the marginal unit of the asset serves both direct and indirect liquidity properties, and this will be reflected in the price. As we move from Region 2 to Region 3, say, because μ increases beyond $\bar{\mu}_{23}$, for some given $A < z(q^*)/2$, the marginal unit of the asset is still providing direct liquidity services, but not indirect.

These results are depicted in Figure 3, where ψ is depicted as a function of μ for two levels of asset supply: $A^h \in (z(q^*)/2, z(q^*))$ and $A^l < z(q^*)/2$. Notice that within the regions where the marginal asset is valued for its liquidity (and, hence, $\psi > \beta$), we also have $\psi'(\mu) > 0$: a higher inflation depresses equilibrium real balances and makes the asset more valuable for its liquidity, regardless of whether this liquidity is direct (as in Regions 2,3) or indirect (as in Region 2). Also, notice that within Region 3 the slope of ψ with respect to μ is the same, regardless of whether $A = A^h$ or $A = A^l$, but, naturally, the equilibrium price is higher under $A = A^l$ because, with a low asset supply, the marginal valuation of agents for the liquidity properties of the asset is higher. The slope of ψ is the highest within Region 2 (which is only relevant if $A = A^l$), because this is precisely where both direct and indirect liquidity kick in.

The model also delivers results concerning the DM production (q) and the trade volume in the OTC, but since these are not of first-order importance for the analysis, we relegate them to Section C of the accompanying Web Appendix.

4 Endogenous Asset Acceptance

Having analyzed equilibrium for any exogenous λ, the task of this section is to determine this important term endogenously. Following LPW, we assume that sellers have the option to pay (*ex ante*) an information cost κ, which allows them to recognize the quality of assets and, consequently, accept them as payment. More generally, one can think of κ as the transaction cost agents must incur so that they can use assets as collateral in private transactions. Clearly, a seller's profit depends on her own choice to acquire the technology/pay the cost that allows her to accept assets, and the decision of other sellers to do so. Let λ denote the representative

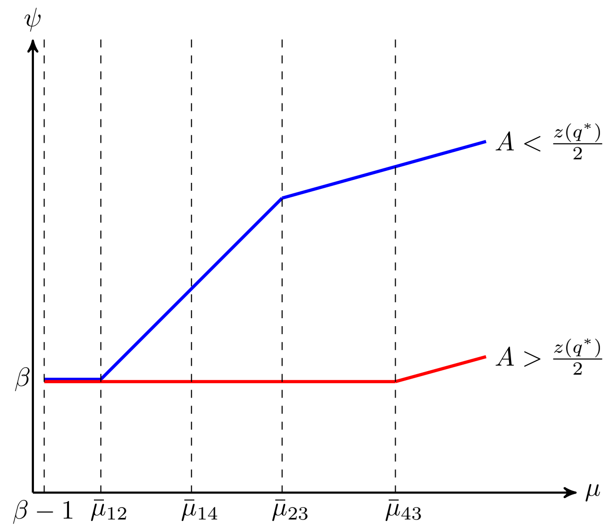

Figure 3: Effects of money growth on the asset price

seller's belief about the measure of (other) sellers that accept assets as payment or collateral. As in Kiyotaki and Wright (1989), we will construct symmetric equilibria where the best response of the representative seller intersects with the 45 degree line.[10]

The *net* profit of a seller who acquires the information, for some $\lambda \in [0, 1]$, is:

$$\Pi(\lambda) \equiv \beta(1-\eta) \Bigg\{ \underbrace{[u(q_2(\lambda)) - q_2(\lambda)]}_{\text{Type 2 Profit}} - \underbrace{\alpha\lambda[u(q_1^m(\lambda)) - q_1^m(\lambda)]}_{\text{Matched Type 1 Profit}} - \underbrace{[1-\alpha\lambda]\,[u(q_1^n(\lambda)) - q_1^n(\lambda)]}_{\text{Not Matched Type 1 Profit}} \Bigg\}.$$

With proportional bargaining, the seller always earns a fraction $1 - \eta$ of the total DM surplus. What is that surplus? If she pays κ, she becomes a type 2 seller and her DM transaction will generate a surplus of $u(q_2) - q_2$. However, by paying κ she gives up the surplus that would have been generated in a type 1 meeting; a surplus that depends on whether the buyer was matched in the OTC (with probability $\alpha\lambda$) or not (with probability $1-\alpha\lambda$). Letting $\Lambda(\lambda)$ denote the seller's optimal response to her belief λ, it is clear that she will choose $\Lambda = 1$, if and only if $\Pi(\lambda) > \kappa$.

Inspection of the definition of Π, reveals that λ affects a seller's profit through two channels. First, it directly affects the probability with which the buyer in a type 1 meeting was matched in

[10] We interpret λ as the *measure* of sellers who accept assets or the *probability* with which sellers accept assets. For any λ, the representative seller will choose her own probability of accepting assets, say Λ, to maximize profits. Any point where this "best response" function intersects with the 45 degree line must be a symmetric equilibrium: it implies optimal behavior (it belongs to the seller's *best response* function) and symmetry (it satisfies $\Lambda = \lambda$).

the OTC. Second, it indirectly affects the DM good traded in the various contingencies, because λ is an important determinant of the demand for the various assets. Studying the properties of $\Pi(\lambda)$ is our main goal in the remainder of this section. We start with an auxiliary lemma.

Lemma 6. *In the steady state equilibrium, and in any possible region, we have* $d\mathrm{w}_1/d\lambda \leq 0$ *and* $d\mathrm{w}_1/d\alpha \leq 0$.

Proof. See the appendix. □

Lemma 6 states that the asset acceptance rate, λ, and the OTC matching efficiency, α, effectively act as inflation. A higher λ induces buyers to carry less money, since they expect to be able to use assets as MOE more often. Likewise, a higher α induces buyers to carry less money, since they expect that it will be easier to get extra cash in the OTC, if they need it. This reveals a complication in characterizing the shape of $\Pi(\lambda)$: if changes in λ mimic changes in μ, the seller who chooses her best response $\Lambda(\lambda)$, must take under consideration that different λ's may be associated with different equilibrium regions; and we know that the various equilibrium q's are different in each region. To see this point, suppose that (A, μ) are indicated by the red dot in Figure 4 (and satisfy $A \in (z(q^*)/2, z(q^*))$ and $\mu < \bar{\mu}_{14}$). Given these parameters, $\lambda = 0$ would imply an equilibrium in Region 1 (upper-left panel). However, as λ increases, the boundaries of the various regions start moving westward, and, eventually, there comes a point where the red dot lies within Region 4 (upper-right panel). Thus, equilibrium now lies in a different region, even though (A, μ) *did not change*. As λ increases further, equilibrium will eventually lie in Region 3 (lower-left panel). Clearly, different parameters would lead to different "paths".[11,12]

With this discussion in mind, we are now ready to study the properties of $\Pi(\lambda)$ and, consecutively, the equilibrium with endogenous λ.

Proposition 2. *a) The derivative of* Π *with respect to* λ *is given by:*

$$\begin{aligned}\Pi'(\lambda) =& \beta[1-\eta]\left\{\frac{\partial q_2}{\partial \lambda}[u'(q_2(\lambda))-1]\right.\\ &-\alpha\Big[[u(q_1^m(\lambda))-q_1^m(\lambda)]-[u(q_1^n(\lambda))-q_1^n(\lambda)]\Big]\\ &\left.-\alpha\lambda\frac{\partial q_1^m}{\partial \lambda}[u'(q_1^m(\lambda))-1]-(1-\alpha\lambda)\frac{\partial q_1^n}{\partial \lambda}[u'(q_1^n(\lambda))-1]\right\}.\end{aligned}$$

[11] For instance, if $A < z(q^*)/2$ and $\mu < \bar{\mu}_{12}$, the red dot would lie in the southwest portion of Region 1 (for $\lambda = 0$), and an increase of λ from 0 to 1 would have brought us through Regions 1, 2, and 3, consecutively. Or, if $A \in (z(q^*)/2, z(q^*))$ and $\mu > \bar{\mu}_{43}$, then equilibrium would lie in Region 3 for any $\lambda \in [0,1]$.

[12] As λ increases, some regions of equilibrium vanish. As we can see in the lower-right panel of Figure 4, the first region to vanish is Region 1. This is intuitive: Region 1 is the region where all types of buyers (except unmatched type 1) get the first best–it is the region of plentifulness. An increase in λ depresses real money balances and makes it impossible for some types to attain q^*. If λ increases further, it is not a surprise that the second region to disappear is the "second most plentiful" region, i.e., Region 4 (that is the region where matched type 1 buyers do not attain q^*, but type 2 buyers do). This is precisely what we see in the lower-right panel of the figure, where the only regions left are 2 and 3. Notice that this plot assumes $\lambda = 1$, but it would look identical for λ in a neighborhood of 1.

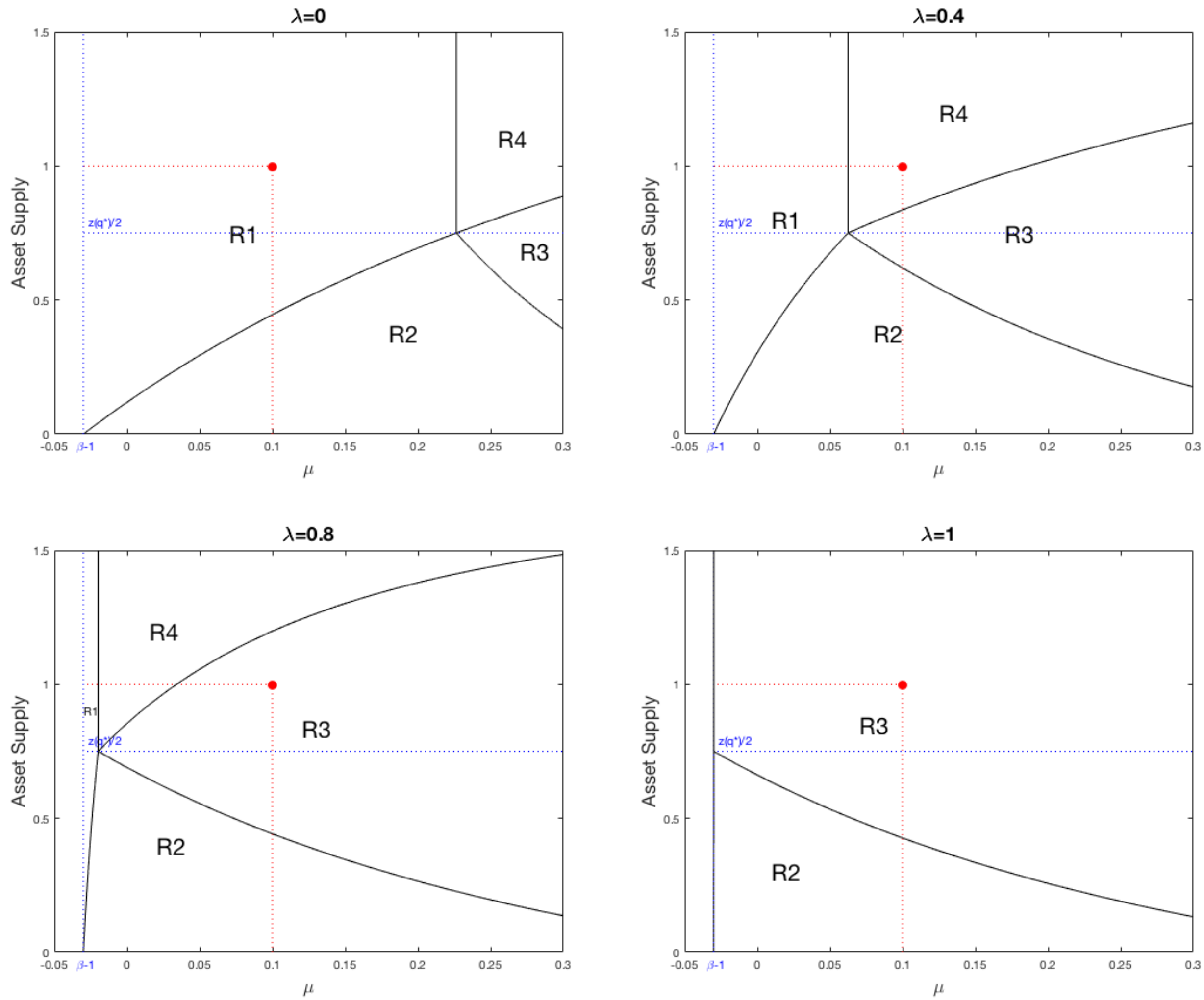

Figure 4: Aggregate regions of equilibrium with different levels of λ. All four cases assume $u(x) = x^{1-\rho}/(1-\rho)$, $\beta = 0.97$, $\rho = 0.5$, $\eta = 0.5$, and $\alpha = 1$.

b) In the steady state equilibrium, a sufficient condition for $\Pi'(\lambda) > 0$ in Region 1 is that u' is log-concave, i.e., $(u'')^2 > u'u''$. In all other regions, the sign of $\Pi'(\lambda)$ is ambiguous. Furthermore, $\partial\Pi/\partial\alpha > 0$ in Region 1 for any parameter values. In all other regions, the sign of $\partial\Pi/\partial\alpha$ is ambiguous.

Proof. See the appendix. □

As we have already discussed, changes in λ affect Π through two channels. Proposition 2 reveals that these channels have opposite directions. On the one hand, a high λ induces buyers to carry few real money balances, as they expect to be able to use their assets as MOE; this channel tends to make Π increasing in λ because a seller who chooses to not get informed has a lot to lose. On the other hand, a high λ implies a high probability of matching for type 1 buyers in the OTC; a seller who did not get informed is very likely to meet a buyer who got matched in the OTC and was, therefore, able to boost her money holdings. This mitigates the loss from not acquiring information. While, the two opposing forces make it difficult to pin down the sign of $\Pi'(\lambda)$ for all parameter values, we are able to show (under slightly stronger assumptions) that $\Pi'(\lambda) > 0$ in Region 1.[13]

The following lemma describes the equilibrium value of λ, which, as already discussed, will be any point where the representative seller's best response function intersects with the 45^o line.

Lemma 7. *The equilibrium value of the λ is as follows:*

1. *When $\Pi(\lambda) < \kappa$, $\forall \lambda \in [0, 1]$, there exists a unique pure strategy Nash equilibrium with $\lambda = 0$.*
2. *When $\Pi(\lambda) > \kappa$, $\forall \lambda \in [0, 1]$, there exists a unique pure strategy Nash equilibrium with $\lambda = 1$.*
3. *If there exist values $\lambda \in [0, 1]$, such that $\Pi(\lambda) = \kappa$, then these values constitute mixed strategy Nash equilibria.*
4. *If multiple mixed strategy Nash equilibria, λ, exist, only those that satisfy $\Pi'(\lambda) < 0$ are stable.*

Proof. The proof is obvious, hence, omitted. □

The interpretation of Lemma 7 is straightforward. An individual seller optimally chooses to become type 2, if $\Pi(\lambda) > \kappa$ for all λ; if this is the case, the unique symmetric equilibrium involves all sellers investing in the technology, i.e., $\lambda = 1$, and the real asset becomes a perfect substitute to money. Clearly, this equilibrium is likely to arise when κ is very small. On the other extreme, if $\Pi(\lambda) < \kappa$ for all λ, the unique equilibrium has no sellers investing, i.e., $\lambda = 0$, and the real asset is fully illiquid. If there exist values of λ for which $\Pi(\lambda) = \kappa$, any $\Lambda \in [0, 1]$ is a best response, hence, such λ's constitute mixed strategy Nash equilibria. We find these "interior" equilibria particularly interesting because they imply that assets are partially liquid, which is the most empirically relevant case. (LPW make a similar argument.)

However, part (4) of Lemma 7 provides a word of caution: only interior equilibria with $\Pi'(\lambda) < 0$ are stable: if an arbitrarily small measure ε of sellers accidentally accept assets, the representative seller's best response is to not accept (i.e., she does not have an incentive to follow the deviant sellers). In contrast, if $\Pi'(\lambda) > 0$, and ε sellers accidentally accepted assets, the individual seller would have an incentive to follow them, thus, "unstabilizing" the equilibrium. Figure 5 illustrates the determination of equilibrium with endogenous λ assuming a

[13] Why is $\Pi(\lambda)$ increasing in Region 1, but not necessarily so in other regions? In Region 1, the first of the two aforementioned channels is quantitatively important, because type 2 buyers can attain q^*, and so type 2 meetings in the DM produce the maximum surplus possible, $u(q^*) - q^*$. Thus, a seller who chooses to not get informed loses a lot. Of course, a seller who does not pay κ will meet with a type 1 buyer, and that buyer's probability of matching in the OTC is increasing in λ. But as discussed earlier in this section, being in Region 1 means that λ is likely to be small anyway. Thus, the relative benefit from not acquiring information is not quantitatively significant. As λ increases, and we move into regions where liquidity is more scarce, the first (positive) force weakens, and the second (negative) force becomes stronger, leading to an ambiguous effect on Π.

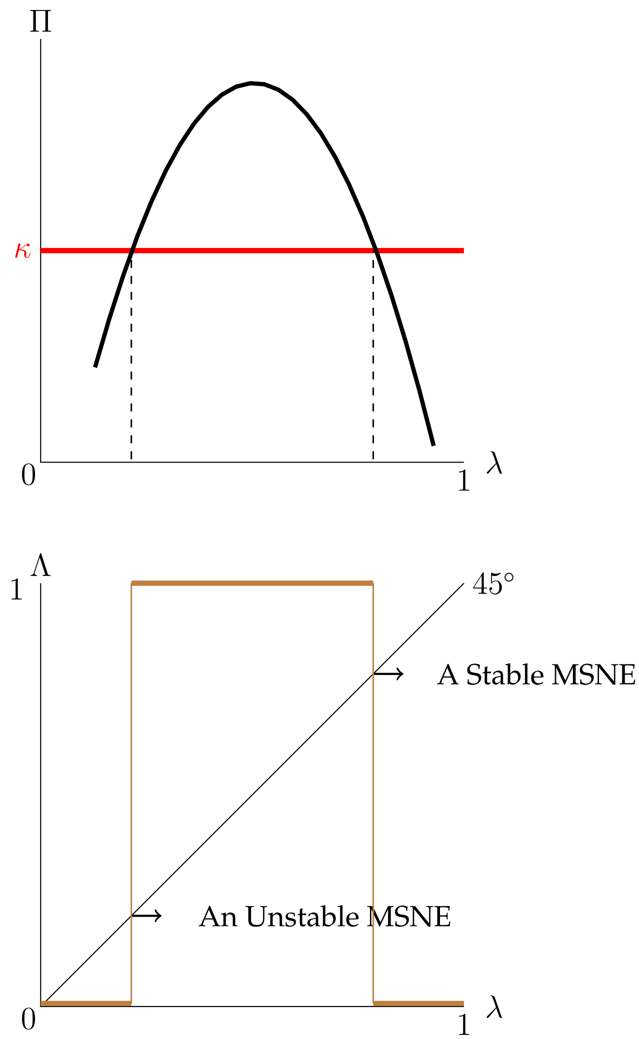

Figure 5: Mixed strategy Nash equilibria for λ when $\Pi(\lambda)$ is hump shaped

hump shaped Π. This example exhibits two interior equilibria, but only the one with the higher λ is stable. Notice that $\lambda = 0$ is also an equilibrium.

The important take-away of this discussion is that our hybrid model of direct and indirect asset liquidity introduces an endogenous force whereby Π *can* be decreasing in λ. This implies that our model can generate stable interior equilibria. Notice that this is not possible in LPW, where Π is a strictly increasing function. LPW also suggest that interior equilibria are the most interesting. It is precisely because the interior equilibrium in that paper is unstable, that the authors explore an extension of the model, where the information cost is different for each seller.[14]

Figure 6 further illustrates the importance of stable interior equilibria. On the left panel, we present the case of a decreasing profit function (and a stable interior equilibrium), which is pos-

[14] With a properly chosen distribution of costs among sellers, LPW can generate stable interior equilibria. We do not wish to claim that heterogeneous costs are unrealistic (perhaps it is quite the opposite). The point made here is that in our model interior equilibria are stable under a wider range of parameters than LPW, because of the novel channel introduced in our framework.

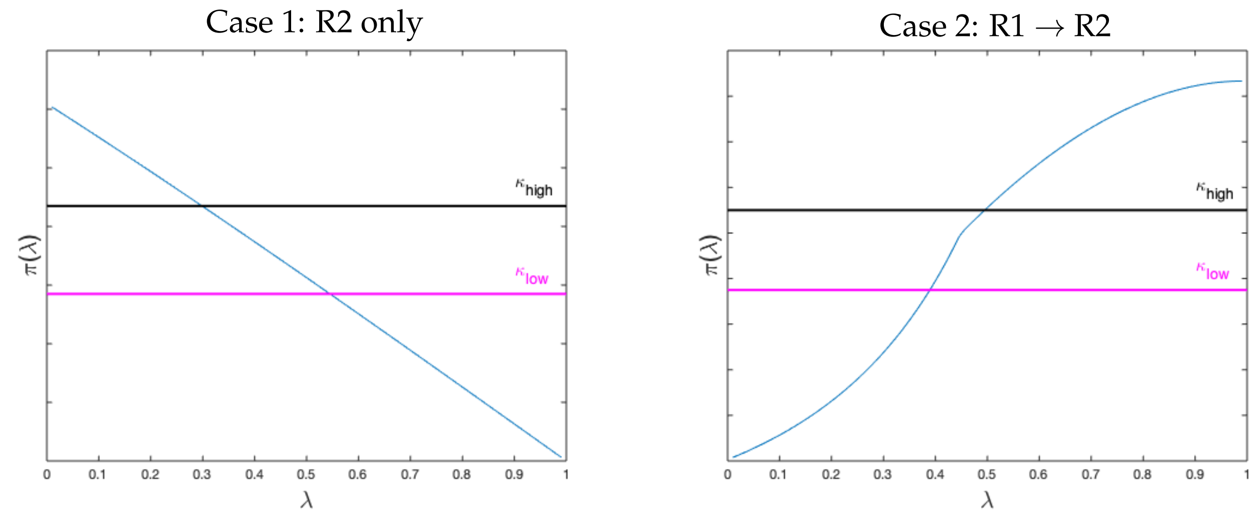

Figure 6: The profit function $\Pi(\lambda)$ under $A < z(q^*)/2$. All cases assume $u(x) = x^{1-\rho}/(1-\rho)$, $\beta = 0.97$, and $\eta = 0.5$. The parameters μ, ρ and A differ, thus, giving Π a different shape in each case. In particular, in Case 1, we have $\mu = 0.001, A = 0.004$, and $\rho = 0.5$; in Case 2, $\mu = 0.004, A = 0.5$, and $\rho = 0.25$.

sible in our framework; on the right panel, we present the case of an increasing profit function (and an unstable interior equilibrium), which is qualitatively equivalent to the model of LPW. Suppose we are interested in how changes in κ affect equilibrium λ. Around the stable interior equilibrium (left panel), an increase in κ would lead to a decrease in the number of sellers who accept assets, as intuition suggests. This is not true for unstable equilibria: on the right panel of Figure 6 an increase in κ would imply a higher λ, a logically inconsistent result. Thus, incorporating the notion of *indirect liquidity* into the model is not only empirically relevant, but also generates a novel channel that improves the theoretical properties of equilibrium.

However, it is important to remind the reader that we do not have an analytical proof that our equilibrium is (unique and) stable. Indeed, Proposition 2 states that $\Pi(\lambda)$ could be increasing, decreasing, or an arbitrary combination of the two, i.e., it states that "anything can happen". Instead of presenting an exhaustive list of all possible equilibria that *could* arise, we choose to describe what *will* happen for a reasonable parametrization of the model. This task is performed in Section 5. For the reader who wishes to review all the possible equilibria that can arise in the model, the detailed analysis is relegated to Section D of the Web Appendix.

5 Numerical Analysis and Equilibrium Welfare

5.1 Calibration

Our objective in this section is to calibrate the model to U.S. data in order to provide a sharper characterization of the Π function. We then use our model to study how changes in the fundamental parameters α and κ affect equilibrium welfare in the model. Following Lagos and Zhang (2020) we set the period length equal to a day. This is consistent with high-frequency OTC trading activity we observe in practice. Time preference is described by the discount factor β. We assume the following functional form for the instantaneous utility: $u(x) = x^{1-\rho}/(1-\rho)$. As in Section 2 the matching function takes the simple constant-return-to-scale (CRS) form $f(x, y) = \alpha xy/(x + y)$, where x, y are the measures of asset buyers and sellers.

The discount rate is set so that $1/\beta$ equals the average daily real interest rate, adopted from Lagos and Zhang (2020). The structural parameters that need to be calibrated are $\{\eta, \lambda, \alpha, A, \rho, \kappa\}$. We set $\{\alpha, \lambda\}$ so that the following two conditions are satisfied. First, following Huber and Kim (2017), we set the matching efficiency $\alpha \in (0, 1)$ so that the matching probabilities of a type 1 buyer and a type 2 buyer in the secondary market are equal. Thus, we have:

$$\delta_1 = \delta_2 \Rightarrow \alpha\lambda = 1 - \alpha\lambda \Rightarrow \alpha = 1/2\lambda. \tag{23}$$

Next, the ratio of OTC trade volume to real balances in the model is given by $\alpha\lambda(1 - \lambda)A/w_1$, where A/w_1 is the ratio of assets outstanding to real balances. Then, the following condition is also true in the model:

$$\alpha\lambda(1 - \lambda) = \frac{\text{OTC trading volume}}{\text{A}}. \tag{24}$$

Combining equations (23) and (24), we obtain:

$$(1 - \lambda) = \frac{\text{OTC trading volume}}{0.5 * A}. \tag{25}$$

Therefore, we match $1 - \lambda$ with the ratio of US Treasury securities outstanding to average daily US Treasury securities trading volume, which is 0.9359 (in 2014).

Next, we match the steady state ψ from our model to the average daily price of 3-month U.S. T-Bills from 2000 to 2019, i.e., $\bar{\psi} = 1/(1 + i) = 0.99996$. Thus, the following equation, reflecting (19), must be satisfied:

$$\bar{\psi} = \beta\Big\{(1 - \lambda)\big[\alpha\lambda L(w_1 + A) + (1 - \alpha\lambda)\big] + \lambda L(w_1 + A)\Big\}, \tag{26}$$

We also match the steady state inflation rate from the model to the average daily U.S. inflation rate adopted from Lagos and Zhang (2020), $\bar{\mu} = 0.000073$. Then, the following condition, reflecting equation (18), should be met as well:

$$\frac{1+\bar{\mu}}{\beta} = \left\{(1-\lambda)\big[\alpha\lambda L(w_1 + A) + (1-\alpha\lambda)L(w_1)\big] + \lambda L(w_1 + A)\right\}. \tag{27}$$

Combining equations (26), (27) with the calibrated values of $\{\lambda, \alpha\}$, gives us $L(w_1 + A)$ and $L(w_1)$.

Next, we set η such that the markup in DM matches with the retail data used by Faig and Jerez (2005), that is, a target markup of 30%. Note that the markup e in the DM is given by

$$\begin{aligned} e =& \frac{\text{Nominal price of a DM good in a type 1 meeting}}{\text{Nominal marginal labor disutility}}, \\ =& \frac{M_t/z^{-1}(w_1)}{1/\varphi_t} = \frac{w_1}{z^{-1}(w_1)}, \text{ where } w_1 = (1-\eta)\frac{[z^{-1}(w_1)]^{1-\rho}}{1-\rho} + \eta z^{-1}(w_1), \end{aligned}$$

which can be rearranged into

$$e = \eta + [z^{-1}(w_1)]^{\rho} - \eta[z^{-1}(w_1)]^{-\rho},$$

where $[z^{-1}(w_1)]^{\rho}$ can be obtained from $L(w_1)$.

For the preference parameter ρ, we follow the standard approach in the monetary economics literature, namely, we set ρ so that the average $L(R) \equiv M/PY$ in the model matches the data.

$$L(R_t) = \frac{w_1}{Y} = \frac{w_1}{(1-\lambda)[\alpha\lambda z^{-1}(w_1 + A) + (1-\alpha\lambda)z^{-1}(w_1)] + \lambda z^{-1}(w_1 + A)} = \frac{M_1}{GDP},$$

where $[z^{-1}(w_1 + A)]^{\rho}$ can be obtained from $L(w_1 + A)$, given by

$$L(w_1 + A) = \frac{u'(z^{-1}(w_1 + A))}{z'(z^{-1}(w_1 + A))} = \frac{d}{dw_1}z^{-1}(w_1 + A)u'(z^{-1}(w_1 + A)) = \frac{1}{e}u'\left(\frac{w_1 + A}{e}\right) = e^{\rho - 1}(w_1 + A)^{-\rho}.$$

The asset supply, A, is backed up by the liquidity constraint in DM

$$A = (1-\eta)\frac{[z^{-1}(w_1 + A)]^{1-\rho}}{1-\rho} + \eta z^{-1}(w_1 + A) - w_1.$$

Of course, in equilibrium κ is equal to $\Pi(\lambda)$ (and λ satisfies equation 25).

Table 1 summarizes the calibration targets and data sources. Calibrated parameters are summarized in Table 2.

Based on the calibrated parameters we compute the profit function $\Pi(\lambda)$, which is illustrated Figure 7. It turns out that all parameter values support a steady state equilibrium within Region 2. Furthermore, we find that the typical seller's profit function is strictly decreasing, which im-

Table 1: Calibration targets

Description	Value	Sources
Average daily real interest rate	0.00004	Lagos and Zhang (2020)
Average daily net inflation rate	0.00007	Lagos and Zhang (2020)
Average daily 3-month U.S. T-bills interest rate	0.00004	FRED
Average money demand M/PY	0.16	FRED
$\frac{\text{US Treasury securities trading volume}}{\text{US Treasury securities outstanding}}$	0.9359	New York FED
Retail sector markup	0.3	Faig and Jerez (2005)

Table 2: Parameter values

Description		Value
Discount factor	β	0.9999
Preference parameter	ρ	0.0544
Matching efficiency	α	7.8069
Asset supply	A	0.1334
Buyer's bargaining power in DM	η	0.001
An information cost	κ	0.01
Fraction of type 2 sellers	λ	0.064

mediately implies that we have a unique and stable interior equilibrium. In our calibration, only 6.4 percent of sellers choose to get informed and accept assets directly as means of payment. In other words, the direct liquidity of the assets is quite low and, intuitively, this happens because the trade volume in the secondary market for Treasuries is very large in the data.

The next task is to use our calibrated model to ask a number of policy-relevant questions.

5.2 Welfare and Policy Implications

Policy makers around the world are concerned about liquidity in asset markets, and often suggest legislations to improve liquidity. Our model shows there are (at least) two ways to do this. First, the authorities could try to increase secondary market efficiency, α. For instance, they could bring forth legislations that promote more efficient/well-networked interdealer markets. Alternatively, they could take actions that reduce the cost that agents need to incur in order to use assets as payment or collateral (i.e., decrease in κ). One may suggest that these two ways of improving liquidity are virtually equivalent, and one should not worry too much about the details. An increase in α (an improvement of indirect asset liquidity) or a decrease in κ (an improvement of direct asset liquidity) should have a similar effect on welfare, and if one had to guess, it would be reasonable to expect this effect to be positive.

To carefully study the validity of such claim, we start by deriving the steady state welfare

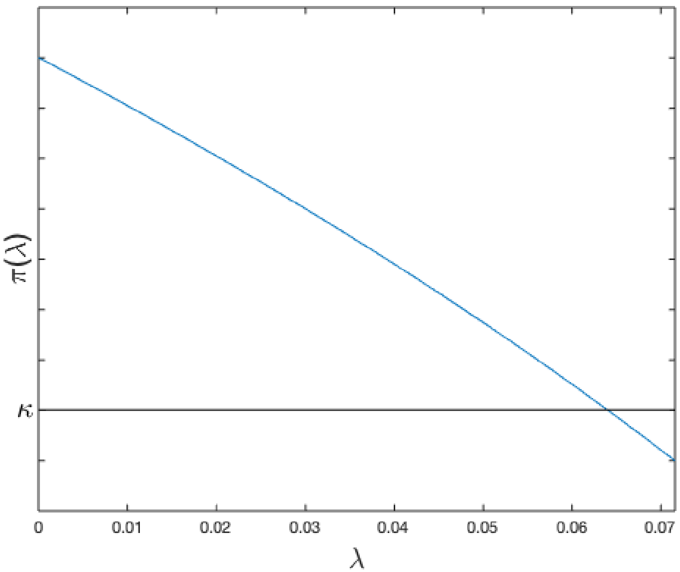

Figure 7: The Profit function $\Pi(\lambda)$

function for this economy, given by

$$\mathcal{W} = \alpha\lambda(1-\lambda)\left[u(q_1^m) - q_1^m\right] + (1-\alpha\lambda)(1-\lambda)\left[u(q_1^n) - q_1^n\right] + \lambda\left[u(q_2) - q_2\right] - \lambda\kappa. \tag{28}$$

The details are relegated to Section E of the Web Appendix, but the intuition is simple: the first two terms in (28) represent the DM surplus generated in type-1 meetings, depending on whether the buyer matched or not in the OTC; the third term stands for the DM surplus in type-2 meetings; the last term is simply the information cost paid by a measure λ of sellers.

First, consider the effect of changes in α on welfare. As seen in the upper-left panel of Figure 8, welfare is monotonically decreasing in α in our calibration. What gives rise to this counter-intuitive result?[15] As indicated in the upper-right panel of the figure, a higher α leads to a lower λ: as the indirect liquidity of the asset improves, the incentive of sellers to pay κ deteriorates, and so does the asset's direct liquidity. Simply put, we have more type-1 buyers in equilibrium. Even though the matching process is more efficient, there are now more type 1-buyers trying to match with fewer type-2 buyers. Whether the matching probability of type-1 buyers in the OTC goes up or down depends on the elasticity of λ with respect to α, but in our calibrated model the total number of matches undoubtedly declines in α (lower-right panel of Figure 8).

In all, a change in α affects directly the OTC matching efficiency, but also indirectly the measure of the various types of meetings. In principle, the sign of $d\mathcal{W}/d\alpha$ can go either way, but

[15] Why counter-intuitive? Because a higher efficiency in the OTC market seems like a positive development. The OTC helps allocate money into the hands of the agents who need it most (i.e., type 1 buyers). Or, in technical terms, a higher α implies that more type-1 buyers match in the OTC, thus, placing a higher weight on the surplus term $u(q_1^m) - q_1^m$, which is greater than $u(q_1^n) - q_1^n$, because matched type-1 buyers have boosted their money holdings.

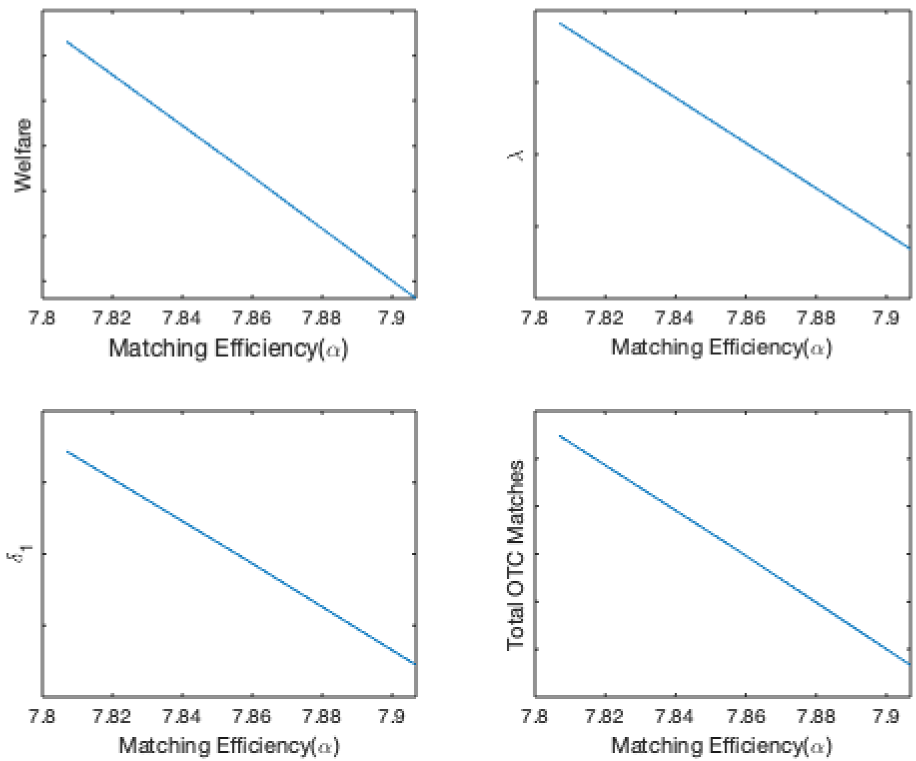

Figure 8: Effects of varying α

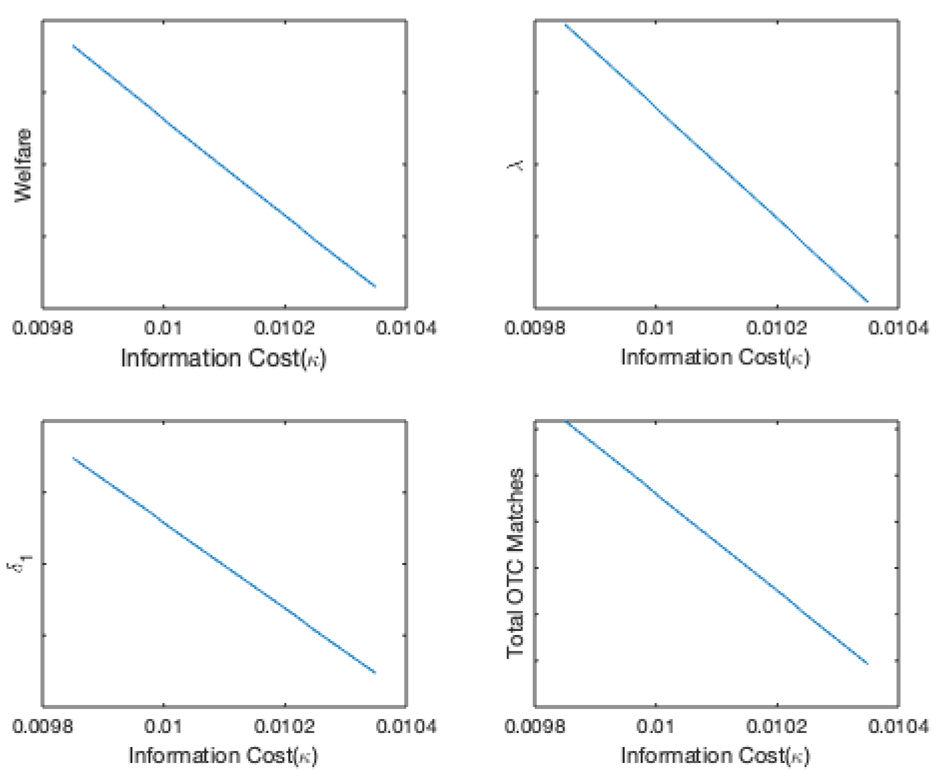

Figure 9: Effects of varying κ

in the calibrated model we find a robust negative relationship. This is yet another illustration of the striking new insights one obtains by adding a secondary market to the model of LPW. (Insights that go beyond the fact that adding such a market is 'realistic'.) The existence of a secondary asset market (in fact, a very efficient one) reduces the ability of the asset to play a direct liquidity role. In equilibrium, we end up with an excess number of buyers seeking additional liquidity in the OTC market, and being unable to find it, even though the efficiency of matching per se has improved.

Next, we consider the effect of changes in κ on welfare. It seems reasonable that a lower κ would increase the fraction of matches where assets serve as collateral (hence, the asset's direct liquidity), and this should be welfare-improving, for two reasons. First, and more obviously, now assets facilitate transactions in more meetings. Second, with a higher λ type-1 buyers are fewer, and they should have an easier time matching in the OTC and proceeding to the DM with more cash (i.e., the exact opposite of what we just saw happening when α increased). This conjecture is confirmed by our calibration, as can be seen in the upper-left panel of Figure 9: indeed equilibrium welfare is higher for lower values of κ.[16]

In sum, an increase in α (effectively an increase in indirect liquidity) and a decrease in κ (effectively an increase in direct liquidity) generate multiple, opposing forces in general equilibrium, and can have very different effects on welfare. Our calibration exercise suggests that authorities have better chances to improve welfare by enhancing the direct liquidity of assets.

A natural question that arises is "what is the real-world counterpart of the model's policy recommendation" (i.e., to improve direct asset liquidity)? One example that fits our model reasonably well is the so-called "eligibility policy", promoted by central banks after the Global Financial Crisis to allow broader classes of assets to serve as collateral and to "improve the pledgeability of assets in private transactions". (See footnote 3 for more details.) More recently, in response to Covid-19, the FED announced the creation of new, and the expansion of existing, facilities to promote the scope of eligible assets for purchase or collateral.[17] Such policies reduce asymmetric information and transaction costs that agents must incur to use assets as collateral and, hence, improve the direct liquidity of assets.

[16] However, once again, there is an opposing force. The higher λ (associated with a lower κ) decreases real balances (Lemma 6), generating a force that tends to reduce welfare. In our quantitative exercise welfare turns out to be strictly decreasing in κ, but one could, in theory, find parameter values for which $d\mathcal{W}/d\kappa > 0$. Generally, $\mathcal{W}$ is likely to have an increasing segment if κ is high and μ is low. This is because, the money demand effect is especially strong when λ is low (that is when we have many type-1 meetings where money is the sole MOE), and this is likely to occur when κ (μ) is high (low). Within this parameter range, a further increase in κ could boost equilibrium real money balances by an amount large enough to generate a positive overall effect on welfare.

[17] Such facilities include the Term Asset-Backed Securities Loan Facility (TALF), the Money Market Mutual Fund Liquidity Facility (MMLF), the Commercial Paper Funding Facility (CPFF) and the Primary Dealer Credit Facility (PDCF). More details about the FED's intervention in March 2020 can be found here:
https://www.sullcrom.com/federal-reserve-new-and-expanded-lending-facilities

5.3 Empirical relationship between direct and indirect liquidity

A key finding of Section 5.2 was that a higher market efficiency, α, leads to a lower equilibrium λ (upper-right panel of Figure 8). Put differently, our model predicts a *negative relationship* between an asset's direct and indirect liquidity. One may be interested to check whether this relationship is empirically supported. To do that one would need to find proxy variables for indirect and direct liquidity and check their relationship; we do so by targeting proxies for these two types of liquidity in U.S. Treasury securities.

Finding a proxy for indirect liquidity is not hard because this notion of liquidity (defined as the ease of trading a security in financial markets) is virtually identical to the notion of 'market liquidity', which is widely studied in finance. The literature offers several candidates to serve as proxies for market liquidity, and we pick one of the most standard ones, namely, bid-ask spreads. More precisely, we will use the inverse of bid-ask spreads for U.S. treasury securities as a proxy for these assets' indirect liquidity. Finding a proxy for direct liquidity is less obvious. Nonetheless, we believe that the notion of 'funding liquidity', which refers to the ease with which an asset can be used as collateral to borrow funds, comes closest to our notion of direct liquidity. Following Brunnermeier and Pedersen (2008), we proxy the funding liquidity of 3-month U.S. treasury bills by the TED spread, defined as the difference between 3-month LIBOR based on U.S. dollars and 3-month U.S. Treasury Bill rates.

To provide some empirical support for our theoretical mechanism, i.e., the negative relationship between direct and indirect liquidity, we would have to show that a positive correlation between bid-ask spreads of 3-month U.S. Treasury bills and the TED spreads of these securities prevails in the data. Indeed, Adrian, Fleming, and Vogt (2017) document such positive correlation. Their regression results show that bid-ask spreads of U.S. Treasury bonds are positively correlated with the TED spread (Figure 2 and Table 7). To provide further support for our mechanism, we carry out our own (analogous) empirical exercise, not only for the U.S., but for four other advanced economies (Canada, Germany, France, and the U.K.). The results are reported in Table 3, where we summarize the correlation between the bid-ask spreads for 3-month government bonds and their TED spreads for each country.[18] Our results in each case indicate a positive correlation between the two proxy variables, and a confirmation of the result obtained by Adrian et al. (2017).[19] We conclude that one of the main mechanisms in our model finds empirical support in the data.

[18] The sample period varies across countries because of data availability, but we extend the sample period as long as possible.

[19] Adrian et al. (2017) focus on Treasury bonds of maturity higher than 2 years. Here we use 3-month Treasury bills. Hence, in the case of the U.S., we do not just repeat the exercise of Adrian et al. (2017), but we also confirm that their result holds true for Treasury securities of shorter maturity.

	Canada	Germany	U.S.	France	U.K.
Correlation Coefficient	0.70***	0.67***	0.32***	0.30***	0.08***
	(-9.906)	(5.452)	(-15.652)	(11.548)	(26.106)
Period	2008.5∼2013.5	2005.10∼2020.1	2001.1∼2020.1	2001.12∼2020.1	2017.7∼2020.1

Table 3: Correlation Between the Bid-Ask Spreads for 3-month Treasury Bills and TED Spreads. T-statistics are in parentheses. *** $p<0.01$, ** $p<0.05$, * $p<0.1$.
Source: Authors' calculation using the data from Datastream, Bloomberg and the central bank website of each country.

6 Conclusion

By now, a large body of literature documents that many assets are priced not only for their fundamental value (roughly defined as the present value of all future payments), but also for the liquidity services these assets may provide. Agents are willing to buy assets at a *liquidity premium* if: i) they expect to use them directly as facilitators of trade (media of exchange or collateral) in transactions; or ii) they expect to sell them for money in a secondary market upon the arrival of a liquidity need. These two types of liquidity, which in our paper have been dubbed "direct" and "indirect", respectively, have been studied extensively in the literature, but only in *isolation*. Our paper demonstrates that this practice is not without loss of generality. Whether a buyer needs to visit a secondary market in order to 'liquidate' assets for cash, depends on whether the seller of the goods/services she wishes to purchase will accept these assets as payment. Vice versa, the willingness of a seller of goods/services to acquire information that allows her to recognize and accept assets (other than money) in transactions, depends on the existence and efficiency of a secondary market where the buyer could sell her assets for cash.

Our model encompasses both of these notions of liquidity and determines their relative importance endogenously, as a function of two fundamental parameters: i) The information cost that sellers must incur in order to recognize and accept assets in transactions; and ii) The efficiency of matching in the secondary asset market. Given these parameters, we study asset prices and how these prices are affected by monetary policy. Our model formalizes some intuitive ideas, but also delivers some new and surprising insights. As an example of the former, we show that a higher secondary market efficiency increases an asset's indirect liquidity and crowds out its direct liquidity, i.e., it discourages sellers from acquiring information and accepting assets as payment. As an example of the latter, we show that changes in direct and indirect liquidity can have very different effects on welfare. An authority whose objective is to maximize welfare has better chances of achieving this goal by promoting direct liquidity of assets, i.e., by taking actions that reduce the transaction costs agents must incur to use them directly as means of payment. On the other hand, a more efficient secondary market can crowd out the asset's direct liquidity to such an extent, that it can ultimately worsen equilibrium welfare.

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A Appendix

Proof. Proof of Lemma 1.

If we take a derivative of the objective function $\eta\{u(q_i) - q_i\}$ in (7) with respect to q_i, the first order condition is that $u'(q_i) = 1$. It implies that the objective function is maximized at $q_i = q^*$ for $i \in \{1, 2\}$. First, consider a Type 1 meeting, where the real balances of a buyer for trade are only determined by his/her money holdings, m. As long as the real money balances, φm, are equal to or greater than $z(q^*)$, q_1 will be equal to the first best quantity, q^*, and the buyer will hand over only m^*, i.e., $d_1^m = m^*$ by the proportional bargaining constraint. It is obvious that $d_1^a = 0$ because a Type 1 seller does not accept real assets as a MOE. On the other hand, the real money balances are strictly less than $z(q^*)$, the buyer gives up all his/her money in order to increase the total surplus, $u(q_1) - q_1$, as much as possible, i.e., $d_1^m = m$, and q_1 is equal to the corresponding amount to the real money balances, $z^{-1}(\varphi m)$. Second, in a Type 2 meeting, the real balances for trade are determined by not only m but also a. Similarly, if the real balances, $\varphi m + a$, are equal to or greater than $z(q^*)$, $q_2 = q^*$, and otherwise $q_2 = z^{-1}(\varphi m + a)$. In the former case, the total real balances that a buyer hand over are exactly equal to $z(q^*)$, but d_2^m and d_2^a are indeterminate only if $\varphi d_2^m + d_2^a = z(q^*)$ by the proportional bargaining constraint because they are perfect substitutes to each other and only the total real value transferred matters. In the latter case, $d_2^m = m$ and $d_2^a = a$ for the same reason in the Type 1 meeting. $\square$

Proof. Proof of Lemma 2.

The OTC bargaining problem simplifies to

$$\max_{\chi,\zeta} S_1 \; s.t. \; S_2 = 0, \;\; \chi \le a, \;\; \zeta \le \tilde{m}, \;\; \text{and } \zeta = p\chi,$$

$$\text{where } S_1 = V_1(m+\zeta, a-\chi) - V_1(m,a) = u(q_1(m+\zeta)) - u(q_1(m)) + \varphi[d_1(m) - d_1(m+\zeta)] + \varphi\zeta - \chi$$
$$= u(q_1(m+\zeta)) - u(q_1(m)) - \chi.$$

The last equality comes from the fact that any trade that would make $m + \zeta > m^*$ would not generate surplus, thereby $d(m) = m$. Finally, S_1 can be expressed as follows.

$$S_1 = u(\tilde{q}_1(m+\zeta)) - u(\tilde{q}_1(m)) - \chi.$$

where $\tilde{q}(m) \equiv \{q : \varphi m = z(q)\}$. Similarly, S_2 can be expressed by

$$\begin{aligned} S_2 =& V_2(\tilde{m} - \zeta, \tilde{a} + \chi) - V_2(\tilde{m}, \tilde{a}) \\ =& u(q_2(\tilde{m} - \zeta, \tilde{a} + \chi)) - u(q_2(\tilde{m}, \tilde{a})) - \varphi\zeta \\ & - \varphi d_2^m(\tilde{m} - \zeta, \tilde{a} + \chi) + \chi - d_2^a(\tilde{m} - \zeta, \tilde{a} + \chi) + \varphi d_2^m(\tilde{m}, \tilde{a}) + d_2^a(\tilde{m}, \tilde{a}). \end{aligned}$$

Now, we will consider two different cases:

Case 1: $\varphi\tilde{m} + \tilde{a} \geq z(q^*)$

In this case, we are in Region 1. Our claim is that post-OTC trade balances must be greater than or equal to $z(q^*)$. We prove this claim by contradiction. Suppose not, i.e., $\varphi\tilde{m}+\tilde{a}+\chi-\varphi\zeta < z(q^*)$. Then, we are in binding DM branch such that

$$\begin{aligned}S_2 =& u(\tilde{q}_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - \varphi\zeta - \varphi\tilde{m} + \varphi\zeta + \chi - \tilde{a} - \chi - u(q^*) + z(q^*) \\ =& [u(\tilde{q}_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - u(q^*)] + [z(q^*) - (\varphi\tilde{m} + \tilde{a})] < 0.\end{aligned}$$

where the last inequality holds true since $\tilde{q}_2 < q^*$ in this region. Thus, it must be that $\varphi\tilde{m} + \tilde{a} + \chi - \varphi\zeta \geq z(q^*)$. Given that we must have $\tilde{q}_2(m+\zeta, a-\chi) = z(q^*)$, $S_2 = \chi - \varphi\zeta$. Thus, any solution must have $\chi = \varphi\zeta$, which also implies $p = 1/\varphi$. Now, the OTC bargaining problem is further simplified to

$$\max_{\zeta} u(\tilde{q}_1(m+\zeta)) - u(\tilde{q}_1(m)) - \varphi\zeta \;\; s.t. \; \zeta \leq \tilde{m}, \;\; \chi \leq a.$$

FOC must be then $u'(\tilde{q})(d\tilde{q}_1/d\zeta) = \varphi$. But since $\varphi m + \varphi\zeta = z(\tilde{q}_1)$, $d\tilde{q}_1/d\zeta = \varphi/z'(\tilde{q}_1)$. Plugging the latter into the FOC should yield $\tilde{q}_1 = q^*$.

Case 2: $\varphi\tilde{m} + \tilde{a} < z(q^*)$

Similar to the case 1, we also claim that $\varphi\tilde{m} + \tilde{a} + \chi - \varphi\zeta < z(q^*)$. Suppose not. Then,

$$\begin{aligned}S_2 =& u(q^*) - z(q^*) - u(\tilde{q}_2(\tilde{m}, \tilde{a})) - \varphi\zeta + \chi + \varphi\tilde{m} + \tilde{a} \\ =& [u(q^*) - u(\tilde{q}_2(\tilde{m}, \tilde{a}))] + [\varphi\tilde{m} - \varphi\zeta + \tilde{a} + \chi - z(q^*)] > 0,\end{aligned}$$

which is a contradiction. So, $\tilde{q}_2 < q^*$ in this case and $S_2 = u(\tilde{q}_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - u(\tilde{q}_2(\tilde{m}, \tilde{a}))$. Since $S_2 = 0$, it must be that $\chi = \varphi\zeta$.

Given solutions in the case 1 and 2, the rest of the proof goes as follows. First, in the case 1 type 1 buyer always wants to set $\tilde{q}_1 = q^*$. Yet, there are 2 reasons why that might not be possible. First, if a is unlimited and $\tilde{m}$ is limited in the sense that $m + \tilde{m} < m^*$, then necessarily $\zeta = \tilde{m}$ and $\chi = \varphi\tilde{m}$. Here, the unlimited a means $a \geq \varphi\tilde{m}$. If $m + \tilde{m} \geq m^*$ and a is limited then, $a = \chi$ and the type 1 cannot get the 1st best, i.e., $a = \varphi\zeta < z(q^*) - \varphi m$. In the case 2, if $m + \tilde{m} \geq m^*$ but $a < z(q^*) - \varphi m$, then $\chi = a$ and $\zeta = a/\varphi$. On the other hand, if $m + \tilde{m} < m^*$ and $a \geq \varphi\tilde{m}$, then $\zeta = \tilde{m}$ and $\chi = \varphi\tilde{m}$. □

Proof. Proof of Lemma 3.

First, we describe how to derive (16) to (22) from (15). We substitute the bargaining solutions in each region in Lemma 2 into (15), and then we take a derivative of J in each region with respect to $\hat{m}$ and $\hat{a}$, respectively, in order to obtain (16) to (22). For example, $\chi = \hat{\varphi}(m^* - \hat{m})$,

$p\chi = m^* - \hat{m}$, and $\rho = z(q^*)$ in Region 1. Plugging these solutions into J yields

$$
\begin{aligned}
J^1(\hat{m},\hat{a}) = & -\varphi\hat{m} - \psi\hat{a} \\
& + \beta\big\{(1-\lambda)\big[\alpha\lambda\big(u(q^*)) + \hat{a} - \hat{\varphi}(m^* - \hat{m})\big) + (1-\alpha\lambda)\big(u(q_1(\hat{m})) + \hat{a}\big)\big] \\
& + \lambda\big[\big(u(q^*)) + \hat{\varphi}\hat{m} + \hat{a} - z(q^*)\big)\big]\big\}.
\end{aligned}
$$

Then it is straightforward to obtain J_1^1 and J_2^1 as show in (16) and (17). We also derive the other derivatives of J in the same way.

Now, we prove the three properties of J mentioned in the lemma.

i. The bargaining solutions and the constraints in the OTC are continuous. Hence, J is continuous. Also, it is differentiable all over the regions across boundaries between regions because J_j for $j \in \{1,2\}$, is continuous across the boundaries: $J_{j+} = J_{j-}$.

ii. J_1 is continuous and strictly decreasing in $\hat{m}$ all over the regions, in particular, because $L(\varphi m)$ is strictly decreasing: $J_1^i < 0$ for $i \in \{1,2,3,4\}$. Hence, it is strictly concave in $\hat{m}$. On the other hand, since J_2 is a constant in Region 1 and 4, and decreasing in $\hat{a}$ in Region 2 and 3: $J_2^i \leq 0$ for $i \in \{1,2,3,4\}$. It is weakly concave in $\hat{a}$.

iii. It is easily proved from i and ii.

□

Proof. Proof of Lemma 4.

Given prices $(\varphi, \hat{\varphi}, \psi)$and beliefs $(\tilde{m}, \tilde{a})$,

i. since J is weakly concave and differentiable everywhere, the condition that $J_j^i(\hat{m},\hat{a}) = 0$ must hold at the optimum;

ii. when $\psi = \beta$, the fact that $L(\cdot) > 0$ in Region 2 and 3 implies that $J_j^i(\hat{m},\hat{a}) > 0$. Hence Region 2 and 3 are ruled out. In Region 1 and 4, J_1^j is strictly concave in $\hat{m}$, and so the optimal choice of $\hat{m}$ is uniquely determined by the condition that $J_1^i(\hat{m},\hat{a}) = 0$ with satisfying the condition that $\varphi > \beta\hat{\varphi}$;

iii. When $\psi > \beta$, Region 1 and 4 are ruled out because the condition that $\psi = \beta$ must hold at the optimum. Moreover, since J_1^j is strictly concave in both $\hat{m}$ and $\hat{a}$ in Region 2 and 3, the optimal portfolio choice, $(\hat{m},\hat{a})$, is uniquely determined the condition that $J_j^i(\hat{m},\hat{a}) = 0$ with satisfying the condition that $\varphi > \beta\hat{\varphi}$.

□

Proof. Proof of Lemma 5.

Since the equilibrium objects $\{q_1^n, q_1^m, q_2^n, q_2^m, \chi, p\}$ are uniquely determined by $\mathrm{w}_1 = \hat{\varphi}(1+\mu)M$, A and ψ, we need to show first that w_1 uniquely exists. Since we only take into account that $\mu > \beta - 1$, and $\varphi > \beta\hat{\varphi}$, an optimal choice of money $\hat{m}$ is uniquely determined by the first order conditions in Lemma 3. Then, φ $(= (1+\mu)\hat{\varphi})$ and $\hat{\varphi}$ should be set such that markets clear, i.e., $\hat{m} = (1+\mu)M$ and $\hat{a} = A$. As a result, w_1 uniquely exists. Moreover, ψ is uniquely pinned down such that the first order conditions in terms of asset holdings in Lemma 3 and $\hat{a} = A$. Hence, $\{q_1^n, q_1^m, q_2^n, q_2^m, \chi, p\}$ also exist and are unique, respectively. □

Proof. Proof of Proposition 1.

Case 1: If $z(q^*) \le A$, the equilibrium is located either in Region 1 or 4, depending on the level of the money growth rate, μ. (17) shows that the asset price equals the fundamental value: $\psi = \beta$.

Case 2: If $z(q^*)/2 < A < z(q^*)$, the equilibrium is in Region 1, 4, or 3.

i. Let $\beta - 1 < \mu \le \bar{\mu}_{43}$. Then, the equilibrium is in Region 1, 4 or the boundary between them, and the asset price, ψ, is equal to β as in Case 1.

ii. Let $\bar{\mu}_{43} < \mu < \infty$. Then, the equilibrium is in Region 3, where $\psi = \beta\{(1-\lambda) + \lambda L(\varphi M + A)\}$. Since $L(\varphi M + A) > 1$ in Region 3, ψ is greater than the fundamental value, β. Moreover, higher μ decreases φM by (20), which leads to higher ψ because $L'(\cdot) < 0$. As a result, ψ strictly increases in μ: $\psi'(\mu) > 0$.

Case 3: If $A < z(q^*)/2$, the equilibrium is in Region 1, 2, or 3.

i. Let $\beta - 1 < \mu \le \bar{\mu}_{12}$. Then, the equilibrium is in Region 1 or the boundary between Region 1 and 2, where the asset price, ψ, is equal to β by (17).

ii. Let $\bar{\mu}_{12} < \mu \le \bar{\mu}_{23}$. Then, the equilibrium is in Region 2 or on the boundary between Regions 2 and 3, where $\psi = \beta\{(1-\lambda)[\alpha\lambda L(\varphi M + A) + (1-\alpha\lambda)] + \lambda L(\varphi M + A)\}$. Since $L(\varphi M + A) > 1$ in Region 2 and its boundary with Region 3, ψ is greater than β. In addition, higher μ decreases φM by (18), which results in higher ψ because $L'(\cdot) < 0$. Hence, ψ strictly increases in μ: $\psi'(\mu) > 0$. Lastly, let $\bar{\mu}_{23} < \mu \le \infty$. Then, the equilibrium is in Region 3, where $\psi(\mu) > \beta$ and $\psi'(\mu) > 0$ as shown in ii. in Case 2.

□

Proof. Proof of Lemma 6

Region 1: From (16) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda + (1-\alpha\lambda)L(\mathrm{w}_1)] + \lambda.$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned} \frac{d\mathrm{w}_1}{d\lambda} &= \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)} \frac{L(\mathrm{w}_1)-1}{L'(\mathrm{w}_1)} < 0, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)\lambda(1-L(\mathrm{w}_1))}{(1-\lambda)(1-\alpha\lambda)L'(\mathrm{w}_1)} < 0. \end{aligned} \tag{a.1}$$

Region 2: From (18) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda L(\mathrm{w}_1 + a) + (1-\alpha\lambda)L(\mathrm{w}_1)] + \lambda L(\mathrm{w}_1 + a).$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned}\frac{d\mathrm{w}_1}{d\lambda} &= \frac{[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]+\alpha(1-\lambda)[L(\mathrm{w}_1+a)-L(\mathrm{w})]-\alpha\lambda[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]}{-(1-\lambda)[\alpha\lambda L'(\mathrm{w}_1+a)+(1-\alpha\lambda)L'(\mathrm{w})]-\lambda L'(\mathrm{w}_1+a)}, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)[\lambda L(\mathrm{w}_1+a)-\lambda L(\mathrm{w}_1)]}{(1-\lambda)[\alpha\lambda L'(\mathrm{w}_1+a)+(1-\alpha\lambda)L'(\mathrm{w}_1)]+\lambda L'(\mathrm{w}_1+a)} < 0.\end{aligned} \tag{a.2}$$

where (a.2) holds true since $L(\mathrm{w}_1+a)-L(\mathrm{w}_1) < \alpha\lambda[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]$.

Region 3: From (20) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda L(2\mathrm{w}_1)+(1-\alpha\lambda)L(\mathrm{w}_1)]+\lambda L(\mathrm{w}_1+a).$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned}\frac{d\mathrm{w}_1}{d\lambda} &= \frac{[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]+\alpha(1-\lambda)[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]-\alpha\lambda[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]}{-(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1)+(1-\alpha\lambda)L'(\mathrm{w}_1)]-\lambda L'(\mathrm{w}_1+a)}, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)[\lambda L(2\mathrm{w}_1)-\lambda L(\mathrm{w}_1)]}{(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1)+(1-\alpha\lambda)L'(\mathrm{w}_1)]+\lambda L'(\mathrm{w}_1+a)} < 0.\end{aligned} \tag{a.3}$$

where (a.3) holds true due to the following. Note that $a > \mathrm{w}_1$ in Region 3. Thus, $L(\mathrm{w}_1+a) < L(2\mathrm{w}_1)$. This leads to $L(\mathrm{w}_1+a)-L(\mathrm{w}_1) < L(2\mathrm{w}_1)-L(\mathrm{w}_1)$. Finally, it must be true that $[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)] < \alpha\lambda[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]$.

Region 4: From (22) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda L(2\mathrm{w}_1)+(1-\alpha\lambda)L(\mathrm{w}_1)]+\lambda.$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned}\frac{d\mathrm{w}_1}{d\lambda} &= \frac{[1-L(\mathrm{w}_1)]+\alpha(1-\lambda)[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]-\alpha\lambda[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]}{-(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1)+(1-\alpha\lambda)L'(\mathrm{w}_1)]}, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)[\lambda L(2\mathrm{w}_1)-\lambda L(\mathrm{w}_1)]}{(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1)+(1-\alpha\lambda)L'(\mathrm{w}_1)]} < .\end{aligned} \tag{a.4}$$

where (a.4) holds since $1-L(\mathrm{w}_1) < \alpha\lambda[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]$.

$\square$

Proof. Proof of Proposition 2

First, we show why $\partial\Pi/\partial\lambda > 0$ in Region 1 under the log-concave utility case. Since we are in Region 1, the following must be true.

$$\Pi'(\lambda) = -\alpha[u(q^*) - q^* - \{u(q_1^n) - q_1^n\}] - (1-\alpha\lambda)[u'(q_1^n) - 1)]\frac{dq_1^n}{d\lambda}, \tag{a.5}$$
$$\frac{dq_1^n}{d\lambda} = \frac{dq_1^n}{d\mathrm{w}_1}\frac{d\mathrm{w}_1}{d\lambda}, \text{ and } \frac{dq_1^n}{d\mathrm{w}_1} = \frac{1}{z'},$$

where the last equality comes from $\mathrm{w}_1 = z(q_1^n)$. Note that we chose to ignore $\beta(1-\eta)$ since it won't affect the result. Combining the above equations with (a.1), one can get

$$\begin{aligned}\frac{dq_1^n}{d\lambda} &= \frac{1}{z'(q_1^n)}\frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{u'(q_1^n)/z'(q_1^n)-1}{L'(\mathrm{w}_1)} = \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{u'(q_1^n)-z'(q_1^n)}{[z'(q_1^n)]^2L'(\mathrm{w}_1)} \\ &= \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{u'(q_1^n)-z'(q_1^n)}{[u''z'-u'z''][1/z']} \\ &= \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{(u'-z')z'}{z'u''-z''u'} = \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{(u'-1)[(1-\eta)u'+\eta]}{u''}.\end{aligned} \tag{a.6}$$

Note that the equality in the second line above comes from $L'(\mathrm{w}_1) = (u''z' - u'z'')/((z')^2)(1/(z'))$, and the 3rd one is from $z' = (1-\eta)u' + \eta,$ and $z'' = (1-\eta)u''$. By combining (a.5) and (a.6), one finally gets

$$\Pi'(\lambda) = \alpha[u(q_1^n) - q_1^n] - \alpha[u(q^*) - q^*] - \frac{1-\alpha\lambda+\alpha(1-\lambda)}{1-\lambda}\frac{[u'(q_1^n)-1]^2[(1-\eta)u'(q_1^n)+\eta]}{u''(q_1^n)},$$

$$H(q) \equiv \alpha[u(q) - q] - \frac{1-\alpha\lambda+\alpha(1-\lambda)}{1-\lambda}\frac{[u'(q)-1]^2\overbrace{[(1-\eta)u'(q)+\eta]}^{z'(q)}}{u''(q_1^n)}.$$

Then $\Pi'(\lambda) = H(q_1^n) - H(q^*)$. Hence, it suffices to show that H is decreasing in q for the proof of $\Pi'(\lambda) > 0$. After some algebra, one can show

$$\begin{aligned}H'(q) = \frac{u'-1}{(1-\lambda)[u'']^2}\Big\{&\alpha(1-\lambda)(u'')^2(1-z') - (1-\alpha\lambda)(u'')^2z' \\ &- [1-\alpha\lambda+\alpha(1-\lambda)][z'\{(u'')^2 - u'u'''\} + (u'-1)(1-\eta)(u'')^2 + z'u''']\Big\} < 0.\end{aligned} \tag{a.7}$$

Inequality (a.7) holds true as long as $(u'')^2 > u'u'''$. This completes the proof.

Next, we prove $\partial\Pi/\partial\alpha > 0$ in Region 1 under any utility functional form. Ignoring $\beta(1-\eta)$, $\partial\Pi/\partial\alpha$ in Region 1 is defined as below.

$$\partial\Pi/\partial\alpha = -\lambda\left[u(q^*) - q^* - (u(q_1^n) - q_1^n)\right] - (1-\alpha\lambda)\left[u'(q_1^n) - 1\right]\frac{dq_1^n}{d\alpha}.$$

Let $G(q) \equiv \lambda(u(q_1^n) - q_1^n) - (1-\alpha\lambda)\left[u'(q_1^n) - 1\right]dq_1^n/d\alpha$. Then, $\partial\Pi/\partial\alpha = G(q) - G(q^*)$. Hence, if $G'(q) < 0$ then $\partial\Pi/\partial\alpha > 0$. One needs to show $\partial\Pi/\partial\alpha > 0$ only for this proof. Similar to (a.6), one can derive the following.

$$\frac{dq_1^n}{d\alpha} = \frac{dq_1^n}{d\mathrm{w}_1}\frac{d\mathrm{w}_1}{d\alpha} = \frac{1}{z'(q_1^n)}\frac{-(1-\lambda)\lambda[1-L(\mathrm{w}_1)]}{(1-\lambda)(1-\alpha\lambda)L'(\mathrm{w}_1)}$$
$$= \frac{-\lambda[z'-u']}{(1-\alpha\lambda)[u''z'-u'z'']}z'$$
$$= \frac{\lambda}{1-\alpha\lambda}\frac{z'(u'-1)}{u''}, \tag{a.8}$$

where the second line in (a.8) follows from $L'(\mathrm{w}_1) = (u''z' - u'z'')/(z')^2(dq_1^n/d\mathrm{w}_1)$, and the third line is from $z'' = u''(1-\eta)$ and $z' = (1-\eta)u' + \eta$. After some algebra along with (a.8), one can obtain:

$$G'(q) = \frac{u'-1}{1-\alpha\lambda}\left\{\lambda(1-\alpha\lambda)\left(1+\eta[u'-1]\right) - \lambda(1-\alpha\lambda)\left(u'+\eta\right)\right\} < 0,$$

where the inequality is true since $1-\eta-\eta < u'(1-\eta)$. This completes the proof. □