Directed Search and the Bertrand Paradox*

Athanasios Geromichalos†
Department of Economics
University of California, Davis

July 2013

Abstract

I study a directed search model of oligopolistic competition, extended to incorporate general capacity constraints, congestion effects, and pricing based on ex-post realized demand. I show that as long as any one of these ingredients is present, the Bertrand paradox will fail to hold. Hence, I argue that, despite the emphasis that has been placed by the literature on sellers' capacity constraints as a resolution to the paradox, the existence of such constraints is only a subcase of a general class of environments where the paradox fails. More precisely, Bertrand's paradox will not arise whenever the buyers' expected utility from visiting a specific seller is decreasing in that seller's realized demand.

JEL Classification: C78; D43; L13

Keywords: Directed Search, Bertrand Paradox, Capacity Constraints, Congestion Effects, State-contingent Pricing

* I am grateful to Giacomo Bonanno for his guidance and assistance. I would also like to thank James Bushnell, Jan Eeckhout, Manolis Galenianos, Philipp Kircher, Demian Pouzo, Martine Quinzii, Dave Rapson, and Kevin Salyer for useful comments and suggestions. All errors are solely my responsibility.

†Contact: ageromich@ucdavis.edu

1 Introduction

In the theory of industrial organization (IO), the *Bertrand paradox* describes a situation in which the competing firms in a duopoly reach a Nash equilibrium where their price equals marginal cost. This result is characterized as a paradox because, typically, one would expect the price to equal marginal cost only in markets with a large number of firms. One resolution to the paradox, that has been emphasized by the theoretical IO literature, is the existence of constraints in the firms' capacity: if a firm sets its price at marginal cost and cannot meet the market demand, the rival firm has an incentive to deviate to a higher price, since it will still be able to attract some customers. In this paper, I show that, despite the emphasis that has been placed by the literature on the capacity constraint resolution, the existence of such constraints is only a subcase of a more general market description where the Bertrand paradox fails to hold. In particular, I demonstrate that Bertrand's paradox is resolved so long as the buyers' expected utility from visiting a certain store is a decreasing function of the realized demand at that store.

I consider a directed search model of oligopolistic competition, where buyers who want to consume one unit of an indivisible good observe the prices of all sellers and visit the one who promises the highest expected utility. I augment the baseline model with three ingredients: capacity constraints, congestion effects, and pricing that depends on ex-post realized demand.[1] I show that, as long as *any one* of these three ingredients is present, the Bertrand paradox ceases to exist, and this is true because the different ingredients share a common feature: they all lead to setups in which a buyer's expected utility from visiting a certain seller depends negatively on how many other buyers show up at that store. As a result, buyers do not necessarily visit the seller with the lowest price, which, in turn, relaxes the price war that typically leads to the Bertrand outcome.

The aforementioned result offers a new way of looking at the role of capacity constraints in the resolution of Bertrand's paradox. With capacity constraints, buyers dislike crowded stores because they are associated with a higher chance of getting rationed. But other things might be going on. Perhaps sellers can serve all the visiting customers, but the buyers' valuation of the good diminishes when the store is crowded. Alternatively, if sellers can price based on ex-post realized demand, they might charge more in the event that many customers show up. In all three scenarios, the paradox breaks down because buyers exert externalities on one another when visiting the same location, thus providing sellers with an incentive to charge higher prices in equilibrium. Somewhat surprisingly, this simple idea has not been formally described before, perhaps because the majority of the existing literature on the Bertrand paradox assumes that sellers face an exogenous demand for their good. On the contrary, in the directed search model, buyers are also strategic agents who realize that their utility from visiting a certain seller depends (for whatever reason) on the realized demand of that seller. Hence, the choice of a directed

[1] A detailed description of these ingredients will follow. In short, the first ingredient means that sellers can produce up to a certain number of units of the good. The second ingredient intends to capture the fact that the consumers' valuation of the good might depend on the number of customers who get served at a certain store (e.g. people tend to dislike crowded restaurants or big lines). The third ingredient allows sellers to charge prices that depend on how many buyers show up at their store.

search model is essential for obtaining the main result of this paper.

In the baseline directed search model, m sellers, usually assumed to possess only one unit of an indivisible good, post prices in order to attract some of the n buyers in the market, and buyers get to visit exactly one seller. The first way in which I depart from the standard model, is by assuming that all sellers can serve up to $k \leq n$ customers, at a constant marginal cost. I treat k as given and describe the symmetric equilibrium price and profit for its various realizations. I show that the equilibrium price exceeds marginal cost if and only if there are *some* capacity constraints, i.e., $k < n$. Moreover, an increase in k implies a higher expected number of sales but, on the other hand, it decreases the local monopoly power of sellers. I show that the symmetric equilibrium price is strictly decreasing in k, and the profit is hump-shaped, eventually reaching zero when $k = n$.

The second way in which I depart from the baseline model, is by assuming that buyers' utility from consuming the good depends negatively on the number of customers served by a certain seller. For instance, consumers tend to dislike crowded stores, restaurants, or amusement parks (too little space or too long waiting lines). In many cases, this type of externality might be more relevant than capacity constraints. For example, companies that provide live streaming of sporting events typically do not ration any customers who want to buy their services. Nevertheless, the quality of streaming for any given customer depends crucially (and negatively) on the total number of customers served. I show that in the presence of such congestion effects, the symmetric equilibrium price will always exceed the marginal cost, *even if sellers face no capacity constraints*. In fact, the more severe the negative externality, the higher the equilibrium price and profit.

The third ingredient that I incorporate into the standard directed search model, is the possibility of pricing based on ex-post realized demand, first studied by Coles and Eeckhout (2003). An example of such pricing behavior is an auction, where the price paid by the buyer ultimately depends on the total number of participants (and their valuation of the good). Assuming that sellers can price based on ex-post realized demand, and that *they do not face capacity constraints*, I show that equilibrium is indeterminate, but there exists a continuum of equilibrium profits that are strictly positive. This indeterminacy of equilibrium has also been documented by Coles and Eeckhout (2003). In their model, each seller has only one unit of the good available. Here, I highlight that this important result also holds true in environments where sellers' capacity is unconstrained.

The three ingredients introduced in the model share a common characteristic: they serve as devices that weaken competition among sellers and help them boost equilibrium profits. However, I show that the welfare properties of equilibrium under the three ingredients are significantly different, and, hence, my model could have some interesting policy implications. Pricing based on ex-post demand can generate high profits, but does not affect the market surplus. In this case, authorities should intervene only if they judge that the sharing rule of the surplus is unfair. On the other hand, if sellers can *collude* on low capacities or on artificially generating congestion, they would pursue these actions, thus achieving higher equilibrium profits but also reducing the total surplus in the market.

Conceptually, the present paper is closely related to Lester (2011), who also revisits a traditional question in economics through the lens of a directed search model. The author studies the relationship between the price setting behavior of sellers and the extent to which consumers can observe these prices before visiting a seller. He shows that the conventional wisdom, according to which in a market with more informed buyers the equilibrium price will be lower, does not necessarily hold in his model. The channel through which this important result emerges, is one that plays a crucial role in my analysis too: in the directed search model, buyers do not necessarily visit the seller with the lowest price.

This paper is related to a large literature on directed search (for example, see Peters (1984), Montgomery (1991), and Burdett, Shi, and Wright (2001)). Many papers in this literature make an extreme, but convenient, assumption on sellers' capacity constraints: they simply assume that sellers possess one unit of the good. Some recent exceptions are Lester (2010), Tan (2010), Watanabe (2010), and Hawkins (2013). The idea that buyers' valuation of the good depends on the number of customers who get served at a certain store is also explored in Geromichalos (2012). There is also a number of papers which study pricing based on ex-post realized demand, *à la* Coles and Eeckhout (2003). Examples of such papers include Julien, Kennes, and King (2000), Eeckhout and Kircher (2010), Virág (2011), and Jacquet and Tan (2012).

Finally, this paper is related to a number of papers, in the theoretical IO literature, which have proposed various resolutions to Bertrand's paradox (and which I do not attempt to fully survey here). One strand of the literature suggests that the paradox fails if sellers interact repeatedly (for example, see Dudey (1992)). Another strand focuses on product differentiation (for example, see Shaked and Sutton (1982)). In static environments with a homogeneous good, the resolution that has attracted the greatest share of attention is the existence of capacity constraints. This idea dates back to Edgeworth (1897). In other notable work, Kreps and Scheinkman (1983) consider a game where sellers choose their capacity and their prices sequentially, and show that the resulting equilibrium coincides with Cournot's (Cournot (1838)) outcome. The present paper attempts to offer a new perspective, by arguing that the existence of capacity constraints is just a special case of a general class of environments where the paradox fails to hold.

In this paper, all sellers (and their goods) are homogeneous, and I focus on symmetric equilbria. Nevertheless, it should be noted that the insights that I provide for the resolution of the Bertrand paradox are not in stark contrast to the idea of differentiated goods.[2] Even if two sellers are identical in every physical way (same good, same technology, etc.), in my model, due to capacity constraints, congestion effects, or state-contingent pricing, a seller who gets visited, on average, by n customers is not the same in the eyes of the consumer as a seller who gets visited, on average, by $n' \neq n$ customers. This, in turn, implies that a seller who deviates to a price that exceeds that of her competitor will not necessarily lose all her customers, which is key for the resolution of the Bertrand paradox.

The rest of this paper is organized as follows. In Section 2, I present a directed search model, extended to incorporate general capacity constraints, congestion effects, and pricing based on ex-post realized demand. Sections 3, 4, and 5 describe equilibrium in environments containing the ingredients described above in isolation. In Section 6, I compare the welfare properties of equilibrium in the three environments and discuss the model's policy implications. Section 7 concludes.

[2] I would like to thank an anonymous referee for pointing out this alternative interpretation.

2 The Model

In this section, in the interest of generality, I describe a model of price posting and consumer search with the following three ingredients: 1) sellers face capacity constraints, 2) buyers' valuation of the good depends negatively on the total number of customers who get served at a certain store, and 3) sellers are able to post prices that depend on ex-post realized demand. In the sections that follow, these three ingredients are not present simultaneously. This is intentional. The purpose is to highlight that only one ingredient is necessary to generate equilibria where the Bertrand paradox fails to hold.

My model builds on Burdett, Shi, and Wright (2001). I consider a market with n buyers and m sellers, $n, m \geq 2$ and finite. All agents are risk neutral. All buyers are identical and anonymous, and each wishes to purchase one unit of an indivisible good. Each seller can produce $i \leq k$ units of the good at a linear cost $c(i) = ci$, $c > 0$. Hence, k is the capacity constraint that sellers face. I treat this parameter as given and describe the pricing decisions of sellers based on the various realizations of k. As it is common in the directed search literature, I assume that buyers can only visit one seller, and they cannot coordinate their visiting strategies.[3] The combination of these two assumptions captures the notion of frictions in the directed search model.

Buyers' utility from consuming the good depends on the number of customers who get served at a certain location: if a seller serves $i \leq n$ customers, the utility enjoyed by each customer is $u(i)$, and for any $h < i$, $u(h) \geq u(i)$. This assumption aims to capture the existence of negative consumption externalities (people dislike to eat in over-crowded restaurants), or congestion effects, such as lines or long waiting times. Furthermore, define $\sigma(i) \equiv i[u(i) - c]$, i.e., the net surplus generated by a seller who serves i customers. I assume that $\sigma(i)$ is non-decreasing for all $i \leq k$. This assumption guarantees that it is optimal for sellers to serve as many buyers as their capacity allows. Hence, rationing will occur only if the number of visiting buyers exceeds k.

The exchange process consists of two stages. At the first stage, given the value of k (which is common for all sellers), each seller posts a price advertisement, taking as given the strategies of her $m - 1$ competitors. A price announcement for seller j is a vector $\mathbf{p}^j = \left(p_1^j, ..., p_n^j\right)$, where p_i^j is the price paid to seller j by customers who get served if that seller gets visited by i buyers. At the second stage, buyers observe all the advertisements and choose a probability of visiting each seller, taking as given the strategies of other buyers. Let i represent the number of buyers who show up at seller j. If $i \leq k$, all buyers get served, but if $i > k$, the seller serves exactly k buyers chosen at random. The buyers who get served enjoy a net utility of $u(\min\{k, i\}) - p_i^j$, and those who get rationed (if any) get a payoff of zero. I assume that sellers commit to their advertisements.

In order to have all agents participate in the trading process, the expected utility and

[3] The assumption that buyers visit only one seller is just an extreme version of the idea that it is costly to visit more than one sellers. The results of this paper (and any other paper that considers a static directed search model) would go through under milder versions of this concept, e.g., buyers who do not get served today can try again tomorrow, but there is discounting. The assumption that buyers cannot coordinate their searches is very standard in the literature, and it will be captured by focusing on mixed strategies by the buyers. For a detailed discussion on this issue, see Burdett, Shi, and Wright (2001).

profit generated by the posted pricing schemes have to be non-negative. Moreover, I assume that buyers can walk away from the trading process at any time and obtain utility equal to zero. Hence, I require that in every contingency, the price paid by buyers who get served cannot exceed their valuation of the good, i.e., for all $i \leq n$, $p_i^j \leq u(\min\{k,i\})$.[4] I refer to these inequalities as *ex-post participation* constraints. I do not impose any assumptions that prevent prices from being smaller than the marginal cost, or even negative in some states, provided that they lead to a non-negative expected profit.

As it is common in the directed search literature, I focus on symmetric equilibria in which buyers play mixed strategies in the subgame. A rich set of equilibria in pure strategies exist, but they are considered implausible since they require an unreasonable degree of coordination among the buyers, in the sense that a buyer needs to know where other buyers are going (see also footnote 3). Equilibria in mixed strategies have gained popularity in the literature, precisely because they are consistent with the frictions that directed search models were designed to capture in the first place.

Consider ex-ante payoffs. Suppose that seller j, who announces $\mathbf{p}^j$, gets visited by an arbitrary buyer with probability θ. The expected utility of a buyer who visits seller j is

$$U^j\left(\mathbf{p}^j,\theta\right) = \sum_{i=1}^{n} \binom{n-1}{i-1} (1-\theta)^{n-i}\,\theta^{\,i-1}\frac{\min\{i,k\}}{i}\left[u\left(\min\{i,k\}\right) - p_i^j\right],$$

where $\binom{n-1}{i-1}$ denotes the binomial coefficient, and the expected profit of seller j is

$$\pi^j\left(\mathbf{p}^j,\theta\right) = \sum_{i=1}^{n} \binom{n}{i}(1-\theta)^{n-i}\theta^{\,i}\min\{i,k\}\left(p_i^j - c\right).$$

For future reference it is useful to define the function

$$H(i,n,\theta) \equiv \binom{n-1}{i-1}(1-\theta)^{n-i}\,\theta^{\,i-1}\frac{1}{i}, \tag{1}$$

which represents the probability with which a buyer who visits seller j gets served, when seller j has only one unit of the good available and a total number of i customers show up. Given the definition in (1) and noticing that $\binom{n}{i} = (n/i)\binom{n-1}{i-1}$, allows one to re-write the expected utility and profit functions above as

$$U^j\left(\mathbf{p}^j,\theta\right) = \sum_{i=1}^{n} H(i,n,\theta)\min\{i,k\}\left[u\left(\min\{i,k\}\right) - p_i^j\right], \tag{2}$$

$$\pi^j\left(\mathbf{p}^j,\theta\right) = n\theta\sum_{i=1}^{n} H(i,n,\theta)\min\{i,k\}\left(p_i^j - c\right). \tag{3}$$

Notice that the term $\sum_{i=1}^{n} H(i,n,\theta)\min\{i,k\}\,p_i^j$ appears in both (2) and (3). This observation will be key for solving the sellers' profit maximization problem.

---

[4] For reasons that will become clear later, in Section 4, I adopt a stronger assumption in order to guarantee existence of equilibrium for all parameter values.

Concluding this section, I present two important algebraic facts, which I will use repeatedly in the paper. Recall the definition of $H(i,n,\theta)$ in (1) and also define the function

$$F(i,n,\theta) \equiv \frac{i-n\theta}{1-\theta}. \tag{4}$$

The following results hold true:

$$\textbf{Fact 1:} \qquad \sum_{i=1}^{n} H(i,n,\theta)\, i = 1,$$

$$\textbf{Fact 2:} \qquad \sum_{i=1}^{n} H(i,n,\theta) F(i,n,\theta)\, i = 1.$$

Fact 1 is obvious.[5] The proof of Fact 2 is relegated to the appendix. Interestingly, although the functions H, F look quite complex when taken individually, the sum of their product, also multiplied by i, is equal to the unit.

3 Capacity Constraints

In this section, I assume that sellers can accommodate only up to $k \leq n$ buyers, and I ignore congestion effects and sophisticated pricing mechanisms based on ex-post demand. As I mentioned earlier, the purpose of this strategy is to stress that only one of the three ingredients of the model described in Section 2 is necessary in order to generate equilibria where the Bertrand paradox is resolved. More precisely, I assume that $u(i) = u > c$, for all $i \leq k$.[6] Also, the typical seller j can only advertise a unique price, $p_i^j = p^j$, for all $i \leq n$. Under this specification, equations (2) and (3) become

$$U^j\left(p^j,\theta\right) = \left(u - p^j\right) \sum_{i=1}^{n} H(i,n,\theta) \min\{i,k\}, \tag{5}$$

$$\pi^j\left(p^j,\theta\right) = n\theta\left(p^j - c\right) \sum_{i=1}^{n} H(i,n,\theta) \min\{i,k\}. \tag{6}$$

The term $\pi^j\left(p^j,\theta\right)$ is the expected profit of seller j, when she announces p^j and gets visited by an arbitrary buyer with probability θ. The term $U^j\left(p^j,\theta\right)$ is the expected utility of a buyer who visits seller j.

The typical seller chooses her price, taking as given the prices of her rival sellers. Since, the goal here is to construct symmetric equilibria, I assume that all other sellers post the (same) price $\tilde{p}$. Seller j sets p^j understanding that, in the second stage, buyers

[5] The expression $H(i,n,\theta)i$ is the probability with which a buyer's preferred seller gets visited by a total of i buyers. Hence, the sum over all possible events must add up to 1.

[6] One of the assumptions adopted in Section 2 required the function σ to be non-decreasing for all $i \leq k$. Here, $\sigma(i) = i(u-c)$ and, therefore, $u > c$ guarantees that $\sigma(i)$ is strictly increasing.

will observe $p^j, \tilde{p}$ and will determine the probabilities with which they visit each seller, such that they are indifferent among all sellers (recall that buyers are assumed to play mixed strategies in the subgame). Hence, if θ represents the probability with which an arbitrary buyer visits seller j, then the probability with which that buyer visits any other seller is given by $\tilde{\theta} = \tilde{\theta}(\theta) = (1-\theta)/(m-1)$.[7] Formally, seller j solves

$$\max_{p^j} \pi^j \left(p^j, \theta\right)$$
$$s.t.\ U^j \left(p^j, \theta\right) = U^l(\tilde{p}, \tilde{\theta}).$$

The term $U^l(\tilde{p}, \tilde{\theta})$ is the expected utility that buyers obtain if they visit any seller $l \neq j$. It is described by (5), if one substitutes p^j with $\tilde{p}$ and θ with $\tilde{\theta}$.

Having established the sellers' problem, I now state the first main result of this section.

Lemma 1. *In the unique symmetric equilibrium, every buyer visits each seller with probability $\theta^* = 1/m$, and all sellers announce the price*

$$p^*(n,m;k) = \frac{\sum_{i=1}^n H\left(i,n,\frac{1}{m}\right)\min\{i,k\}\left\{\frac{m}{m-1}\left[1-F\left(i,n,\frac{1}{m}\right)\right]u + F\left(i,n,\frac{1}{m}\right)c\right\}}{\frac{1}{m-1}\sum_{i=1}^n H\left(i,n,\frac{1}{m}\right)\min\{i,k\}\left[m-F\left(i,n,\frac{1}{m}\right)\right]}, \quad (7)$$

where the functions H and F are defined in (1) and (4), respectively.

Proof. A key observation is that the term $p^j \sum_{i=1}^n H(i,n,\theta)\min\{i,k\}$ appears in both $\pi^j(p^j,\theta)$ and $U^j(p^j,\theta)$. Solving the constraint in the seller's problem with respect to this term and substituting into the profit, reveals that seller j's objective function is

$$\max_{\theta} n\theta \left[(u-c)\sum_{i=1}^n H(i,n,\theta)\min\{i,k\} - U^l(\tilde{p},\tilde{\theta})\right].$$

Hence, the seller's objective can be written as a function of the variable θ only. The first-order condition for seller j's problem will be necessary and sufficient (even though here $k \in \{1,...,n\}$ the proof is very similar to Burdett, Shi, and Wright (2001), where $k=1$). This condition is given by

$$(u-c)\sum_{i=1}^n H(i,n,\theta)\min\{i,k\} - U^l(\tilde{p},\tilde{\theta}) +$$
$$+\theta\left[(u-c)\sum_{i=1}^n \frac{\partial H(i,n,\theta)}{\partial\theta}\min\{i,k\} - (u-\tilde{p})\sum_{i=1}^n \frac{\partial H(i,n,\tilde{\theta})}{\partial\tilde{\theta}}\frac{d\tilde{\theta}}{d\theta}\min\{i,k\}\right] = 0. \quad (8)$$

[7] Notice that the analysis is consistent with the fact that the market studied here is an *oligopoly*. The number of sellers, m, is small enough so that when a seller chooses her price, she understands that this will affect not only the probability with which buyers visit her store in the second stage, but also the probability with which buyers visit her rivals. If m is very large, the effect of p^j on $\tilde{\theta}$ is so small that can be safely ignored. This is equivalent to the *market utility* approach often employed in the literature. Under this approach, sellers maximize profits subject to the constraint of providing visiting buyers with a certain level of utility, which they take as given. Examples of papers which build on this method are Montgomery (1991), Lang (1991), Acemoglu and Shimer (1999), and Galenianos and Kircher (2009).

Notice a few important points. First, $d\tilde{\theta}/d\theta = -1/(m-1)$. Second,

$$\frac{\partial H(i,n,x)}{\partial x} = H(i,n,x)\left(\frac{i-1}{x} - \frac{n-i}{1-x}\right).$$

Third, since the focus is on symmetric equilibria, one can now (after obtaining the first-order condition) impose symmetry: all sellers must post the same price, $p^j = \tilde{p} = p^*$, which, in turn, implies that all buyers will visit each seller with the same probability, $\theta = \tilde{\theta} = \theta^* = 1/m$. Once these three observations are incorporated into (8), we are left with one equation in one unknown, p^*. Solving with respect to this term, after some manipulations, yields the desired result. □

Lemma 1 provides a closed form solution for the unique symmetric equilibrium price, for any n, m, k. Naturally, p^* is increasing in n and decreasing in m. More interestingly, p^* is strictly decreasing in the capacity of sellers. Intuitively, given the frictions in the environment (inability of buyers to coordinate), a lower k implies a higher probability of rationing. This provides sellers with a greater local monopoly power, and allows them to charge a higher per unit price. The monotonicity of p^* in k is an important finding, which, to my knowledge, has not been shown in any other paper in the directed search literature.[8] Since the proof contains some technical details that are not essential for the understanding of the model, it is relegated to the appendix.

I now focus on the equilibrium profits and state the main result of this section. From (6), in the symmetric equilibrium, each seller has an expected profit equal to

$$\pi^*(n,m;k) = \frac{n}{m}\left[p^*(n,m;k) - c\right]\sum_{i=1}^{n} H\left(i,n,\frac{1}{m}\right)\min\{i,k\}. \tag{9}$$

Since every term inside the summation is positive, sellers make a positive profit if and only if p^* is greater than the marginal cost. The next proposition establishes that this is indeed the case as long as sellers face some capacity constraints.

Proposition 1. *a) If $k = n$, then $p^*(n,m;k) = c$ and, hence, $\pi^*(n,m;k) = 0$.*
b) If $k < n$, then $p^*(n,m;k) > c$ and, hence, $\pi^*(n,m;k) > 0$.

Proof. a) For the economy of space, I write $H(i)$ for $H(i,n,1/m)$ and $F(i)$ for $F(i,n,1/m)$. First, notice that if $k = n$, one can write

$$\begin{aligned} p^*(n,m;k) &= \frac{\sum_{i=1}^{n} H(i)i\left\{\frac{m}{m-1}\left[1 - F(i)\right]u + F(i)c\right\}}{\frac{1}{m-1}\sum_{i=1}^{n} H(i)i\left[m - F(i)\right]} = \\ &= \frac{\frac{m}{m-1}u\sum_{i=1}^{n} H(i)i[1-F(i)] + c\sum_{i=1}^{n} H(i)iF(i)}{\frac{1}{m-1}\left[m\sum_{i=1}^{n} H(i)i - \sum_{i=1}^{n} H(i)F(i)i\right]} = c, \end{aligned}$$

[8] As mentioned in the Introduction, most papers in the directed search literature assume that $k = 1$.

where the last equality follows directly from Facts 1 and 2. Thus, if $k = n$, the equilibrium profit for sellers is zero, and the Bertrand result emerges.

b) To see why any $k < n$ is sufficient for $p^*(n, m; k) > c$, recall that $p^*(n, m; k)$ is strictly decreasing in k, for all $k = 1, ..., n$ (see proof in the appendix). In other words, for all $k = 1, ..., n-1$, $p^*(n, m; k) > p^*(n, m; k+1)$. Since $p^*(n, m; n) = c$, we have that $p^*(n, m; k) > c$ for all $k < n$. □

Proposition 1 contains the most important result of this section. Sellers will achieve a strictly positive profit in the symmetric equilibrium, if and only if they face *some capacity constraints*, i.e., $k < n$. The equilibrium profit is the product of the mark-up term $p^*(n, m; k) - c$ and the term $(n/m) \sum_i H(i) \min\{i, k\}$, which is the expected number of sales (equation (9)). Clearly, expected sales are increasing in k and, as discussed earlier, $p^*(n, m; k)$ is decreasing. Hence, an increase on k has two opposing effects on the equilibrium profit. Typically, an increase in k has a very large (positive) effect on the number of expected sales when k is small, but this effect becomes very small for k close to n.[9] As a result, $\pi^*(n, m; k)$ is increasing for small values of k, and eventually it decreases and reaches 0 when $k = n$ (part (a) of the proposition).

One exception to the hump-shaped profit occurs when the market tightness n/m is small. In this case, the increase in expected sales that will follow a rise of k from 1 to 2 is already very small because, with a small n/m ratio, even if $k = 1$ the sellers do not ration many customers (in expected terms). Thus, the negative effect of increasing k (a lower price) is dominant even for very small values of k, and the symmetric equilibrium profit is maximized for $k = 1$. For example, suppose that $u = 1, c = 0.2, n = 5, m = 3$. Then, for $k = 1, ..., 5$, we obtain $\pi^*(1) = 0.493, \pi^*(2) = 0.438, \pi^*(3) = 0.156, \pi^*(4) = 0.019$ (and, of course, $\pi^*(5) = 0$), so that $\pi^*(k)$ is strictly decreasing in k throughout its domain.

4 Consumption Externalities

In this section, I assume that sellers can accommodate all the buyers who visit their store, i.e., $k = n$. Also, as in the previous section, sellers post a unique price, $p_i^j = p^j$, for all $i \leq n$. The interesting feature of this section is that buyers' valuation of the good depends on the number of customers who get served at their preferred location. If the seller serves i customers, the utility enjoyed by each is given by $u(i)$, and I assume that for any $h < i$, $u(h) \geq u(i)$. Hence, this environment is characterized by negative consumption externalities or, alternatively, congestion effects.

In this section (only), I relax the ex-post participation assumption, that is, I do not assume that the posted price should be no greater than the buyer's valuation in every

[9] To see this point through an example, suppose that $m = 2, n = 10$, and let $q(k)$ denote the expected number of sales (per seller) in the symmetric equilibrium. It can be easily shown that $q(1) = 0.999, q(2) = 1.988$, and that $q(9) = 4.999, q(10) = 5$. Hence, an increase in k from 1 to 2, increases the expected number of sales by almost 100%, but an increase in k from 9 to 10 has essentially no effect on expected sales, since the seller was already guaranteed to capture half of the market (in expected terms).

single contingency. The reason is straightforward. Here, $k = n$, $u(i)$ is non-increasing in i, and sellers can only post one price. If n is large and $u(n)$ is strictly decreasing (say $u(i) = 1/i$), there will always exist events in which the buyer's valuation for the good is tiny. Hence, it might be impossible to find a single equilibrium price, p^*, that satisfies $p^* \leq u(i)$ for very large i's. To guarantee existence of equilibrium, I will only require that the posted prices lead to a non-negative expected utility for the buyers. In some sense, this is equivalent to assuming that the buyer commits to paying the posted price at the location she picked. Notice that doing so is actually optimal, if the buyer is assumed to not know the total number of visiting customers when she makes the payment.[10]

Under this specification, and using Fact 1, one can re-write equations (2),(3) as[11]

$$U^j\left(p^j, \theta\right) = \sum_{i=1}^{n} H(i, n, \theta) i u(i) - p^j, \tag{10}$$

$$\pi^j\left(p^j, \theta\right) = n\theta\left(p^j - c\right), \tag{11}$$

The interpretation of these terms is as in Section 3. Suppose that all sellers but j post $\tilde{p}$, and let $\tilde{\theta} = (1-\theta)/(m-1)$ denote the probability with which the arbitrary buyer visits each of these sellers. Then, seller j solves

$$\max_{p^j} \pi^j\left(p^j, \theta\right)$$
$$s.t.\ U^j\left(p^j, \theta\right) = U^l(\tilde{p}, \tilde{\theta}).$$

The next lemma describes the equilibrium price.

Lemma 2. *In the unique symmetric equilibrium, every buyer visits each seller with probability $\theta^* = 1/m$, and all sellers announce the price*

$$p^*(n, m) = c + \frac{m}{m-1} \sum_{i=1}^{n} H\left(i, n, \frac{1}{m}\right) i \left[1 - F\left(i, n, \frac{1}{m}\right)\right] u(i), \tag{12}$$

where the functions H and F are defined in (1) and (4), respectively.

Proof. The proof follows the same steps as that of Lemma 1. One can replace p^j from the constraint of the seller's problem into the profit function, and re-write seller j's objective only as a function of the probability θ,

$$\max_{\theta} n\theta \Big\{ \sum_{i=1}^{n} [H(i, n, \theta) - H(i, n, \tilde{\theta})] i\, u(i) + \tilde{p} - c \Big\}.$$

[10] Relaxing the ex-post participation assumption guarantees existence of a symmetric equilibrium, but it does not affect the properties of this equilibrium. Put differently, if this assumption is maintained, existence of equilibrium is not guaranteed for all parameter values. However, when equilibrium does exist, it still satisfies the property highlighted in the forthcoming Proposition 2, i.e., $p^* > c$. Hence, relaxing the ex-post participation requirement does not change the nature of the main result.

[11] Since here $k = n$, the buyer pays p^j with certainty, and her net expected utility depends on the total number of buyers who visit seller j. The seller's profit is equal to the number of expected sales, $n\theta$, times the per unit mark-up $p^j - c$.

Taking the first-order condition with respect to θ, imposing the usual conditions of symmetry, and solving with respect to the price yields[12]

$$p^*(n,m) = \frac{\sum_{i=1}^n H\left(i,n,\frac{1}{m}\right) i \left\{ \frac{m}{m-1}\left[1 - F\left(i,n,\frac{1}{m}\right)\right] u(i) + F\left(i,n,\frac{1}{m}\right) c \right\}}{\frac{1}{m-1}\sum_{i=1}^n H\left(i,n,\frac{1}{m}\right) i \left[m - F\left(i,n,\frac{1}{m}\right)\right]}.$$

The formula reported in (12) follows from using Facts 1 and 2 in the expression above. □

Having provided a closed form solution for the equilibrium price for any n, m and for any function $u(i)$, I now describe the symmetric equilibrium profit, $\pi^*(n,m)$.

Proposition 2. *a) If $u(i) = u$ for all i, then $p^*(n,m) = c$ and, therefore, $\pi^*(n,m) = 0$.*
b) If for all $i \in \{1, ..., n-1\}$, $u(i) \geq u(i+1)$, with strict inequality for some i, then $p^*(n,m) > c$ and, therefore, $\pi^*(n,m) > 0$.

Proof. It is understood that $H(i) = H(i,n,1/m)$ and $F(i) = F(i,n,1/m)$.
a) If $u(i) = u$, one can write

$$p^*(n,m) = c + \frac{m}{m-1} u \sum_{i=1}^n H(i)\, i \left[1 - F(i)\right] = c,$$

since, from Facts 1 and 2, $\sum_{i=1}^n H(i)\, i\left[1 - F(i)\right] = 0$.
b) If $u(i) \neq u$, $p^*(n,m)$ is given by (12). As pointed out earlier, $\sum_{i=1}^n H(i)i[1 - F(i)] = 0$. Moreover, the term $1 - F(i)$ is decreasing in i, all $i \leq n$, and satisfies $1 - F(1) = (n-1)/(m-1) > 0$, and $1 - F(n) = 1 - n < 0$. Therefore, as long as $u(i)$ is decreasing (in the precise sense that $u(i) \geq u(i+1)$, with strict inequality for some i), multiplying the terms $H(i)i[1 - F(i)]$ with $u(i)$, $i = 1, ..., n$, assigns greater weights on the relatively greater values. Since $\sum_{i=1}^n H(i)i[1 - F(i)] = 0$, it must be that $\sum_{i=1}^n H(i)i[1 - F(i)]u(i) > 0$. Then, from (12), $p^*(n,m) > c$ follows immediately. □

Proposition 2 indicates that when the buyers' valuation of the good depends on the total number of customers who visit a certain location, sellers can achieve positive profits in the symmetric equilibrium, even in the absence of any capacity constraints. To my knowledge, this result has not been documented in the directed search (or any other) literature. It highlights that sellers can use the state dependent valuation of the good as a collusion device in order to increase equilibrium profits. In Section 6, I discuss some potential policy implications of this finding.

I now illustrate Proposition 2 using a concrete example. Let $u(i) = v - \kappa(i-1)$, with $\kappa > 0$ and $v > c$. Hence, the buyer's valuation decreases linearly in the number of other buyers that visit the same store. To guarantee that $\sigma(i) = i[v - c - \kappa(i-1)]$ is

[12] As in Section 3, the first-order condition will be necessary and sufficient. Mathematically the term $u(i)$ has the same effect on the seller's objective function as the term $\min\{i,k\}$ in Section 3; it puts non-increasing weights on terms associated with higher i's (i.e., when the seller gets visited by more customers).

non-decreasing, I impose the restriction $\kappa \leq (v-c)/(2n-1)$. Under this specification, the equilibrium price becomes

$$p^*(n,m) = c + \kappa \frac{m}{m-1} \sum_{i=1}^{n} H\left(i,n,\frac{1}{m}\right)\left[F\left(i,n,\frac{1}{m}\right)-1\right] i^2.$$

Moreover, one can show that $\sum_{i=1}^{n} H\left(i,n,1/m\right)\left[F\left(i,n,1/m\right)-1\right] i^2 = (n-1)/m$ (see Fact 3 in the proof of Lemma 3 in the appendix), implying that

$$p^*(n,m) \;\; = \;\; c + \kappa \, \frac{n-1}{m-1}.$$

Clearly, the equilibrium price and the profit, $\pi^* = \kappa[n(n-1)][m(m-1)]^{-1}$, are increasing in the parameter κ, which captures the reduction in the valuation of the good when one additional buyer visits a certain store. When κ is large, the expected utility from visiting a store with many customers is diminished. Thus, buyers are willing to choose a seller with a higher price, hoping that this seller will be visited by fewer customers. This, in turn, gives an incentive to sellers to post higher prices. Of course, in symmetric equilibrium, all sellers get the same number of expected buyers. However, the existence of congestion effects serves as a collusion device allowing sellers to achieve positive profit, even if $k = n$.

5 Pricing Based on Ex-Post Realized Demand

In this section, I maintain the assumption $k = n$. Also, as in Section 3, I assume that $u(i) = u > c$, for all $i \leq n$. The interesting results of this section are driven by the fact that sellers can post prices that are contingent on ex-post realized demand. An example of such pricing behavior is an auction, where the price paid by the buyer critically depends on the total number of participants: if a buyer is the only participant (and this is public info), she will pay the reservation price. But if many buyers show up, they will bid the price up to their valuation.[13] As another example, the London-based internet cafes which belong to easyGroup have been using a pricing system which they refer to as variable pricing. According to this system, the price that the cafe charges fluctuates depending how busy the store is.

In the model, I assume that the typical seller j can advertise a vector $\mathbf{p}^j = \left(p_1^j, ..., p_n^j\right)$, where p_i^j is the price paid to seller j by each customer when i customers show up. To

[13] It should be pointed out that the equilibrium outcome of an auction will coincide with the one just described, only if the sellers face some capacity constraints (this is the case in Coles and Eeckhout (2003)). Since in this section there are no capacity constraints, one could argue that the auction interpretation does not perfectly fit the ex-post pricing mechanism considered here. This is indeed true, but only because of my strategy to focus on the three ingredients of my model in isolation (for emphasis). If I allowed some capacity constraints together with ex-post pricing, then the auction would be an applicable example to motivate this type of pricing.

facilitate comparison with Sections 3 and 4, here I wish to develop an environment where a buyer's expected utility from visiting a certain seller is decreasing in the number of visiting customers (as was the case in these sections). To that end, I focus on pricing schemes that are increasing in the number of visiting buyers. That is, I assume that the posted prices satisfy $p_k^j \leq p_l^j$, for $k < l$. Moreover, in order to obtain a sharp characterization of equilibrium, I restrict attention to linear pricing schemes, $p_i^j = a^j + b^j(i-1)$, with $b^j \geq 0$, where it is understood that a^j is the price paid by a customer who was the only one to visit seller j. Ex-post participation of the buyers requires that $p_n^j = a^j + b^j(n-1) \leq u$. Prices in some states can be smaller than the marginal cost, or even negative (and sellers are committed to honor these announcements), as long as they lead to a non-negative expected profit. Under this specification, equations (2) and (3) become[14]

$$U^j\left(\mathbf{p}^j, \theta\right) = u - \sum_{i=1}^{n} H(i, n, \theta)\, i\, p_i^j, \tag{13}$$

$$\pi^j\left(\mathbf{p}^j, \theta\right) = n\theta \left[\sum_{i=1}^{n} H(i, n, \theta)\, i\, p_i^j - c\right], \tag{14}$$

where $p_i^j = a^j + b^j(i-1)$. The interpretation of these terms is as in Section 3. As in previous sections, suppose that all sellers but j post $\tilde{\mathbf{p}} = (\tilde{p}_1, ..., \tilde{p}_n)$, where $\tilde{p}_i = \tilde{a} + \tilde{b}(i-1)$, $i = 1, ..., n$. Also, let $\tilde{\theta} = (1-\theta)/(m-1)$ denote the probability with which the arbitrary buyer visits each of these sellers. Then, seller j solves

$$\max_{\mathbf{p}^j} \pi^j\left(\mathbf{p}^j, \theta\right)$$
$$s.t.\ U^j\left(\mathbf{p}^j, \theta\right) = U^l(\tilde{\mathbf{p}}, \tilde{\theta}).$$

The next lemma describes equilibrium prices.

Lemma 3. *Every price vector $\boldsymbol{p}^* = (p_1^*, ..., p_n^*)$, where p_i^* is given by $p_i^* = a^* + b^*(i-1)$, and where (a^*, b^*) satisfy*

$$a^* = c - b^* \frac{(n-1)(m-2)}{m(m-1)}, \tag{15}$$

$$a^* \leq u - b^*(n-1), \tag{16}$$

together with a strategy for the buyers to visit each seller with probability $\theta^* = 1/m$, constitutes a symmetric equilibrium.

Proof. Since the proof contains some technical details that are not essential for the understanding of the model, it is relegated to the appendix. □

[14] In this environment, buyers enjoy utility u with certainty, and they pay a price that depends on the total number of customers who visit seller j. This is in contrast with Section 4 (equation (10)), where buyers pay a certain price, and the valuation of the good depends on ex-post realized demand.

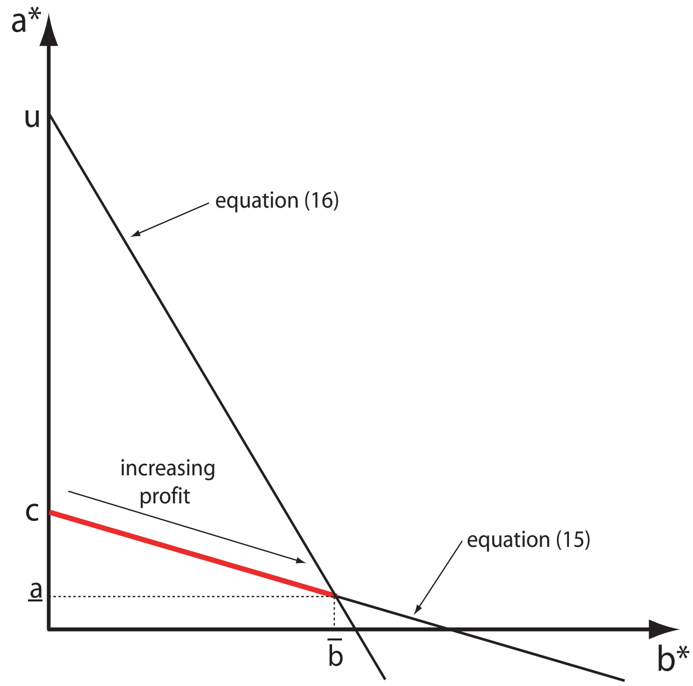

Figure 1: All (a^*, b^*) indicated by the red line are consistent with equilibrium.

Lemma 3 reveals that the equilibrium prices are not uniquely determined. The set of all possible equilibrium prices is illustrated by the bold red line in Figure 1. Every price scheme $p_i^* = a^* + b^*(i-1)$, where (a^*, b^*) satisfy (15) and do not violate ex-post participation (i.e., the inequality in (16)), is part of a symmetric equilibrium.[15] This result is in the spirit of Coles and Eeckhout (2003). However, in that paper, $k = 1$. This paper demonstrates that the well-known indeterminacy result, documented by Coles and Eeckhout (2003), also holds true in environments without capacity constraints. To my knowledge, this result has not been pointed out in the literature. I will return to explain the intuition behind this finding after describing equilibrium profits in the following proposition.

Proposition 3. *a) If sellers cannot apply state-contingent pricing, then the unique symmetric equilibrium has* $p_i^* = c$ *for all* $i = \{1, ..., n\}$ *and equilibrium profit* $\pi^*(n, m; \boldsymbol{p}^*) = 0$.

b) If sellers can apply state-contingent pricing, there exists a continuum of positive profit levels that can be supported in the symmetric equilibrium.

Proof. a) When the typical seller can only post $p_i^j = p^j$, for all $i = 1, ..., n$, equation (14) implies that this seller's profit is equal to $n\theta(p^j - c)$. At the same time, equation (13) implies that the expected utility of a buyer who visits that seller is equal to $u - p^j$ (that is, it does not depend on θ). This is just a version of the standard Bertrand game, and

[15] Notice that the duopoly is an interesting special case. With $m = 2$, we must always have $a^* = c$, and equilibrium prices are indexed by $b^* \in [0, (u-c)/(n-1)]$.

it is straightforward to show that the unique equilibrium of this game has $p^* = c$, which implies that the equilibrium profit is zero.

b) The result is an immediate consequence of the price indeterminacy. As I show in the appendix, the symmetric equilibrium profit is given by

$$\pi^* (n, m; \mathbf{p}^*) = \frac{n}{m}\left[a^* - c + b^* \frac{n-1}{m}\right].$$

Since the pairs (a^*, b^*) that are consistent with equilibrium satisfy (15), the equilibrium profit can be re-written as

$$\pi^* (n, m; \mathbf{p}^*) = b^* \frac{n(n-1)}{m^2(m-1)}. \tag{17}$$

Moreover, ex-post participation will be satisfied as long as $b^* \in [0, \bar{b}]$, where I have defined $\bar{b} \equiv m(m-1)[(n-1)(m^2-2m+2)]^{-1}(u-c) > 0$. The term $\bar{b}$ is derived in the appendix and depicted in Figure 1. Hence, the equilibrium profit is indexed by the various values of b^* in the set $[0, \bar{b}]$. With the exception of $b^* = 0$, all these equilibria are associated with strictly positive profits. □

Proposition 3 indicates that, in the absence of state-contingent pricing, the unique equilibrium satisfies $p^* = c$, and the Bertrand paradox emerges (part (a)). If sellers can post prices based on ex-post realized demand, a continuum of equilibria with strictly positive profits exist, and the various equilibria are not payoff equivalent. To see this point, fix the strategy of all sellers but j, and consider this seller's best response. When seller j has access to a pricing scheme p_i^j, she can advertise more buyer surplus in some states and less in other states (by changing a^j, b^j in the opposite direction), leaving expected payoffs of other players unaffected. One way to see this point, is to notice that, although the seller chooses a^j, b^j, her best response can be summarized by a choice of only one variable, i.e., the variable θ (this is equation (a.6) in the appendix). Intuitively, given any price announcement by her rivals, seller j has a best response correspondence (rather than function), which, by symmetry, gives rise to a continuum of equilibrium prices. More importantly, in the small market, the various best responses are associated with different levels of expected utility in the subgame (since market utility is not fixed), which implies that the sharing rule of the surplus is not uniquely pinned down.

It follows from equation (17) that the equilibrium profit is increasing in b^*. Clearly, the best equilibrium for buyers is the Bertrand equilibrium, with $(a^*, b^*) = (c, 0)$. On the other hand, the preferred equilibrium for sellers is the one associated with $b^* = \bar{b}$. In this case, it is easy to verify that $a^* = \underline{a} \equiv u - m(m-1)(m^2-2m+2)^{-1}(u-c)$.[16] It should be noted that even in this equilibrium (the sellers' optimal), sellers cannot extract the whole surplus in the market. To see this point, let $U^* (n, m; \mathbf{p}^*)$ denote the symmetric equilibrium expected utility of the typical buyer, and notice that sellers will extract all the

[16] In Figure 1, the parametric specification is such that $\underline{a} > 0$. However, obtaining a negative $\underline{a}$ is also a possibility. Typically, this will happen if the ratio u/c is very large.

market surplus, only if $U^*(n, m; \mathbf{p}^*) = 0$, which requires that $p_i^* = u$, for all i. However, from part (a) of Proposition 3, we know that any fixed price schedule, not only will not deliver $U^*(n, m; \mathbf{p}^*) = 0$, but, in fact, it will lead to the extreme opposite, i.e., a situation where $\pi^*(n, m; \mathbf{p}^*) = 0$.

Another interesting implication of Proposition 3 and, in particular, equation (17) is that the only parameters that affect equilibrium profit are n, m (in a positive and a negative fashion, respectively). The parameters u and c do not appear in (17). This might be surprising at first. However, recall that the various equilibrium profits are indexed by $b^* \in [0, \bar{b}]$, and notice that $\bar{b}$ depends positively on the term $u - c$. Hence, even though u, c do not appear directly in (17), they critically affect the range of equilibrium profits that can be sustained: the higher the distance $u - c$, the higher the profits that are consistent with equilibrium.

6 Comparison of the Three Environments

In Sections 3, 4, and 5, I illustrated how the directed search model, augmented with capacity constraints, congestion effects, or state-contingent pricing, respectively, can generate equilibria where the Bertrand paradox does not emerge. In this section, I highlight that these different ingredients lead to the same result (failure of the Bertrand paradox), because they share a common feature. Moreover, I compare equilibrium welfare under the three environments and discuss potential policy implications of my model.

From Section 2, recall equation (2), which I repeat here for convenience,

$$U^j\left(\mathbf{p}^j, \theta\right) = \sum_{i=1}^{n} H(i, n, \theta) \min\{i, k\} \left[u\left(\min\{i, k\}\right) - p_i^j\right].$$

This is the expected utility of a buyer who visits seller j, when that seller announces a price schedule $\mathbf{p}^j$ and gets visited by an arbitrary buyer with probability θ. If there are no capacity constraints, then $\min\{i, k\} = i$ for all $i \leq n$. If there are no congestion effects, then $u\left(\min\{i, k\}\right) = u$. Finally, if seller j cannot post state-contingent prices, then $p_i^j = p^j$. In an environment where these three scenarios hold true simultaneously, equation (2) becomes $U^j\left(\mathbf{p}^j, \theta\right) = u - p^j$. In words, buyers get served with certainty at every location, and they pay the announced price. As a result, sellers who announce prices higher than the competition get no customers. This leads to a price war that will end only when all sellers set $p^* = c$, so that the Bertrand equilibrium arises.

It is now clear that the three different ingredients introduced in the model share a common feature. in the presence of *any* of the three ingredients, buyers tend to dislike sellers with many customers: when $k < n$, many customers could imply rationing; when there are congestion effects, the valuation of the good diminishes in crowded stores; and when sellers can post state-contingent prices, they might charge more when ex-post realized demand is high. In all cases, buyers are willing to visit sellers with higher prices because these sellers will tend to have fewer customers. Sellers accept the offer and indeed charge higher prices in equilibrium. To use the game theory jargon, the existence of (any of) the three ingredients studied in this paper, serves as a *collusion device* that allows sellers to achieve positive profits.

I believe that this is an important finding, given that the IO literature has emphasized the existence of capacity constraints as the most prominent resolution to the Bertrand paradox (at least in markets with homogeneous sellers and good). This paper highlights that the existence of capacity constraints is just a subcase of a more general market description in which the Bertrand paradox fails to hold: equilibrium profits will be positive in markets where the buyers' expected utility from visiting a seller is a (decreasing) function of that seller's ex-post realized demand. This negative relationship (between expected utility and the number of customers) will be satisfied if sellers face capacity constraints, but it will also be satisfied if there are congestion effects and/or if sellers have access to state-contingent pricing.

I now examine the welfare properties of equilibria under the three environments. Define the expected total surplus, $\mathcal{S}^* \equiv nU^* + m\pi^*$, where π^* is expected profit per seller, and U^* is expected net utility per buyer, in the symmetric equilibrium. One can show that in the model with capacity constraints, congestion effects, and state-contingent pricing, respectively, the expected total surplus is given by

$$\mathcal{S}^*(n,m;k) = n(u-c)\sum_{i=1}^{n} H\left(i,n,\frac{1}{m}\right)\min\{i,k\}, \tag{18}$$

$$\mathcal{S}^*(n,m;u) = n\sum_{i=1}^{n} H\left(i,n,\frac{1}{m}\right) i\left[u(i)-c\right], \tag{19}$$

$$\mathcal{S}^*(n,m;\mathbf{p}^*) = n(u-c). \tag{20}$$

In the model with capacity constraints, every match generates a surplus equal to $u-c$, but matches are not guaranteed. Since $\sum_{i=1}^{n} H\left(i,n,1/m\right)\min\{i,k\}$ is the probability with which a buyer gets served in the symmetric equilibrium, multiplying that term with n yields the total number of expected matches in the economy (or the matching function). Clearly, $\mathcal{S}^*(n,m;k)$ is strictly increasing in k, for all $k \leq n$. In the model with congestion effects, all buyers get served. A buyer finds herself at a store with a total of i customers with probability $H(i,n,1/m)i$. In this event, a surplus $u(i)-c$ per person is generated. Stronger congestion effects imply a lower value of $\mathcal{S}^*(n,m;u)$. Finally, in the model with state-contingent pricing, all buyers get served, and their valuation of the good does not depend on the number of visiting customers (although the price they pay does).

Most modern economies enforce antitrust laws whose objective is to protect customers and prevent anti-competitive practices, most notably *price fixing*.[17] But this paper highlights that sellers do not need to fix prices in order to boost profits above the Bertrand level. They have other ways to achieve this goal, and, critically, these alternative methods of achieving high-profit equilibria are less likely to be attacked by the antitrust authorities. Moreover, equations (18)-(20) reveal that the three practices, identified in this paper,

[17] Other practices that antitrust laws aim to prevent include formation of cartels, barriers to entry, bid rigging, product bundling, exclusive dealing, and misuse of patents and copyrights.

which help sellers sustain prices above marginal cost have different consequences on social welfare. Hence, this model has some important policy implications.

Consider first state-contingent pricing. As the analysis of Section 5 indicates, this practice can lead to (a continuum of) equilibria in which sellers achieve positive profits. However, equation (20) implies that endowing sellers with the ability to apply ex-post pricing does not affect the social welfare of the economy. Therefore, in this environment, the authorities should intervene only if they judge that the sharing rule of the surplus is unfair, e.g. sellers are making excessive profits.

Now consider the model with capacity constraints. The total surplus, $\mathcal{S}^*$, is strictly increasing in k. However, the discussion in Section 3 reveals that there exists a unique $\iota < n$, such that $\pi^*(n, m; k)$ is decreasing in k, for all $k \geq \iota$. Hence, if sellers can *collude on low capacities*, they will not choose the socially optimal, $k = n$. As an example, consider a market with $m = 5, n = 50, u = 1, c = 0.2$, and write $\pi^*(k)$ and $\mathcal{S}^*(k)$ as shortcuts for $\pi^*(50, 5; k)$ and $\mathcal{S}^*(50, 5; k)$, respectively. Suppose that sellers can silently agree on a value of k, and then, given that k, they legitimately compete over prices.[18] If sellers set $k = 7$, the per seller equilibrium profit is maximized and equals $\pi^*(7) = 4.692$. At the same time, $\mathcal{S}^*(7) = 27.29$. If the authorities could enforce an increase of k by just one unit, the surplus would increase to $\mathcal{S}^*(8) = 30.53$, but this would lead to lower profits, $\pi^*(8) = 4.690$. Similarly, if $k = 9$, then $\mathcal{S}^*(9) = 33.30$ but $\pi^*(9) = 4.352$, and if $k = 10$, $\mathcal{S}^*(10) = 35.525$ but $\pi^*(10) = 3.727$.

Finally, consider the policy implications for the model with congestion effects. For concreteness, consider the numerical example introduced in Section 4, i.e., assume that $u(i) = v - \kappa(i-1)$, with $\kappa > 0$, $v > c$, and $\kappa \leq (v-c)/(2n-1)$. Under this specification, the expected total surplus becomes

$$\mathcal{S}^*(n, m; u) = n \sum_{i=1}^{n} H\left(i, n, \frac{1}{m}\right) i \left[v - \kappa(i-1) - c\right] = \frac{n[m(v-c) - \kappa(n-1)]}{m}.$$

This expression is decreasing in the size of the congestion effect, κ. However, recall from the discussion in Section 4, that $\pi^* = \kappa[n(n-1)][m(m-1)]^{-1}$, which is increasing in κ. Hence, in this market, there is a clear conflict: sellers are better off when κ is large, but social welfare is maximized when $\kappa = 0$. The authorities should be concerned about this conflict, since sellers have a clear incentive to agree (i.e., collude) on practices that increase κ and, thus, decrease $\mathcal{S}^*$ artificially (a comment similar to that in footnote 18 applies here). Such practices could include allowing big queues at the stores or hiring too few employees who cannot provide quality services when many customers show up.

[18] Notice that this setup is consistent with the fact that antitrust laws are heavily focused on collusion practices that involve price fixing. In the scenario described here, sellers *first* silently agree on a low k (critically, they can do so because such a practice is less likely to be attacked by antitrust laws), and *then*, given a low k, they legitimately compete over prices (they are more likely to compete legitimately over prices because collusion on prices is more often targeted by antitrust authorities).

7 Conclusions

In this paper, I revisit the Bertrand paradox through the lens of a directed search model. I augment the baseline model with three ingredients, capacity constraints, consumption externalities, and state-contingent pricing, in isolation, and I show that in each case the Bertrand paradox does not arise. The three different ingredients share a common feature: they make a buyer's expected utility from visiting a specific seller a decreasing function of that seller's ex-post realized demand. In all three cases, buyers dislike stores with many customers and, thus, are willing to visit sellers with higher prices. This economic force leads to equilibria where sellers indeed post prices higher than the marginal cost and achieve strictly positive profits. The directed search model offers a new perspective of looking at the Bertrand paradox, by highlighting that the existence of capacity constraints is just a subcase of a more general class of environments where the paradox fails to hold.

A Appendix

Proof of Fact 2. Notice that for any $i = 1, ..., n$

$$\frac{\partial[H(i,n,\theta)i]}{\partial\theta} = H(i,n,\theta)\left(\frac{i-1}{\theta} - \frac{n-i}{1-\theta}\right) i = \frac{1}{\theta}H(i,n,\theta)[F(i,n,\theta)-1]\, i.$$

Therefore,

$$\frac{1}{\theta}\sum_{i=1}^{n} H(i,n,\theta)[F(i,n,\theta)-1]\, i = \sum_{i=1}^{n}\frac{\partial[H(i,n,\theta)i]}{\partial\theta} = \frac{\partial\left[\sum_{i=1}^{n} H(i,n,\theta)i\right]}{\partial\theta} = 0,$$

where the last equality follows from the fact that $\sum_{i=1}^{n} H(i,n,\theta)i = 1$ (Fact 1). The equation above implies that

$$\sum_{i=1}^{n} H(i,n,\theta)F(i,n,\theta)\, i = \sum_{i=1}^{n} H(i,n,\theta)\, i = 1.$$

□

Proof of the fact that $p^*(n,m;k)$ is strictly decreasing in k. For the economy of space, I write $H(i)$ for $H(i,n,1/m)$ and $F(i)$ for $F(i,n,1/m)$. I show that for all $k = 1, ..., n-1$, $Dp^*(k) \equiv p^*(n,m;k) - p^*(n,m;k+1) > 0$. To start, add and subtract the term $m(m-1)^{-1}c\sum_{i=1}^{n} H(i)\min\{i,k\}[1-F(i)]$ in the numerator of p^* in (7). This allows one to re-write the equilibrium price as

$$p^*(n,m;k) = c + m(u-c)\frac{\sum_{i=1}^{n} H(i)\min\{i,k\}[1-F(i)]}{\sum_{i=1}^{n} H(i)\min\{i,k\}[m-F(i)]}. \tag{a.1}$$

Using (a.1), for any $k < n-1$,

$$Dp^*(k) = \frac{m\,(u-c)\,\Omega}{\sum_{i=1}^n H\,(i)\min\{i,k\}[m-F(i)]\sum_{i=1}^n H\,(i)\min\{i,k+1\}[m-F(i)]}, \qquad \text{(a.2)}$$

where I have defined the term

$$\begin{aligned}\Omega \;\equiv\;& \sum_{i=1}^n H\,(i)\min\{i,k\}[1-F(i)]\sum_{i=1}^n H\,(i)\min\{i,k+1\}[m-F(i)]\\ -\;& \sum_{i=1}^n H\,(i)\min\{i,k+1\}[1-F(i)]\sum_{i=1}^n H\,(i)\min\{i,k\}[m-F(i)].\end{aligned}$$

The proof of monotonicity of p^* will be complete, if I can show that the denominator of the expression on the right-hand side of (a.2) and the term Ω share the same sign. I claim that they are both positive, and I prove this claim below.

Claim 1: Both summations in the denominator of (a.2) are positive. I show the result in detail for the first term. The proof is identical for the second term. Since $m \geq 2$, and $H\,(i) > 0$ for all $i = 1, ..., n$, it is true that $\sum_{i=1}^n H\,(i)\min\{i,k\}[m-F(i)] > \sum_{i=1}^n H\,(i)\min\{i,k\}[1-F(i)]$. I will now show that $\sum_{i=1}^n H\,(i)\min\{i,k\}[1-F(i)] > 0$.

It follows from Facts 1,2 that $\sum_{i=1}^n H(i)i[1-F(i)] = 0$. Moreover, the function $1-F(i)$ is strictly decreasing in i, for all $i \leq n$, and satisfies $1-F(1) = (n-1)/(m-1) > 0$, and $1-F(n) = 1-n < 0$. Hence, there exists a unique $\nu \in \{1, ..., n-1\}$, such that $H(i)i[1-F(i)] \geq 0$ *iff* $i \leq \nu$, and one can write

$$\sum_{i=1}^{\nu} H(i)i[1-F(i)] = -\sum_{i=\nu+1}^{n} H(i)i[1-F(i)]. \qquad \text{(a.3)}$$

Assume, without loss of generality, that $k \geq \nu$ and notice that *all* terms on the right-hand side of (a.3) are positive. This allows me to write

$$\begin{gathered}\sum_{i=1}^{\nu} H(i)i[1-F(i)] > -\sum_{i=\nu+1}^{k} H(i)i[1-F(i)] - k\sum_{i=k+1}^{n} H(i)[1-F(i)] \Leftrightarrow\\ \sum_{i=1}^{n} H(i)[1-F(i)]\min\{i,k\} > 0.\end{gathered}$$

This concludes the proof of the first claim.[19]

Claim 2: Ω is positive. After a series of manipulations, one can obtain

$$\Omega = (m-1)\left[\sum_{i=1}^{k} H\,(i)\,i\sum_{i=k+1}^{n} H\,(i)\,F(i) - \sum_{i=1}^{k} H\,(i)\,F(i)i\sum_{i=k+1}^{n} H\,(i)\right],$$

[19] The intuition of this proof can be summarized as follows. Since the summation (for $i = 1$ to n) of the terms $H(i)i[1-F(i)]$ adds up to zero, and the terms associated with larger values of i are negative, the expression $\sum_{i=1}^n H(i)[1-F(i)]\min\{i,k\}$ tends to assign a smaller "weight" on the negative terms, thus adding up to a value bigger than zero.

and it suffices to show that

$$\sum_{i=1}^{k} H\left(i\right) i \sum_{i=k+1}^{n} H\left(i\right) F(i) > \sum_{i=1}^{k} H\left(i\right) F(i) i \sum_{i=k+1}^{n} H\left(i\right).$$

After multiplying throughout with i, the last inequality becomes

$$\sum_{i=1}^{k} H\left(i\right) i \sum_{i=k+1}^{n} H\left(i\right) F(i) i > \sum_{i=1}^{k} H\left(i\right) F(i) i \sum_{i=k+1}^{n} H\left(i\right) i. \tag{a.4}$$

Finally, use Facts 1 and 2 to replace $\sum_{i=k+1}^{n} H\left(i\right) F(i) i$ and $\sum_{i=k+1}^{n} H\left(i\right) i$ in (a.4). My original claim, that $\Omega > 0$, will hold if and only if

$$\sum_{i=1}^{k} H\left(i\right) i[1 - F(i)] > 0.$$

This inequality holds for every parameter value, since $\sum_{i=1}^{n} H(i) i[1 - F(i)] = 0$, and the function $1 - F(i)$ is strictly decreasing in i. Hence, the proof is complete. □

Proof Lemma 3. **Step 1:** For this proof, it is important to recall Facts 1 and 2. Also, I will introduce one more useful result.

Fact 3: $$\sum_{i=1}^{n} H(i,n,\theta)[F(i,n,\theta) - 1]\, i^2 = (n-1)\theta.$$

To see why Fact 3 is true, recall from the proof of Fact 2 that the functions H, F, have an interesting link. In particular, we saw that

$$\frac{\partial[H(i,n,\theta)]}{\partial\theta} = \frac{1}{\theta} H(i,n,\theta)[F(i,n,\theta) - 1]. \tag{a.5}$$

Therefore,

$$\frac{1}{\theta}\sum_{i=1}^{n} H(i,n,\theta)[F(i,n,\theta) - 1]\, i^2 = \sum_{i=1}^{n} \frac{\partial[H(i,n,\theta) i^2]}{\partial\theta} = \frac{\partial\left[\sum_{i=1}^{n} H(i,n,\theta) i^2\right]}{\partial\theta} = n - 1,$$

where the last equality follows from the fact that $\sum_{i=1}^{n} H(i,n,\theta)\, i^2 = 1 + (n-1)\theta$.[20] Now Fact 3 follows immediately.

[20] Showing this last statement is straightforward. Consider a situation where, say, the n-th buyer visits seller j with certainty, and she believes that the other $n-1$ buyers visit that seller with probability θ. From equation (1), it is clear that the term H is the probability with which the n-th buyer gets served, when seller j has only one unit of the good available and a total number of i customers (including herself) show up. Then, arguing in a similar fashion, the expression $H(i,n,\theta)\, i^2$ is the number of sales for seller j, if that seller never rations buyers (as in Section 5), and she gets visited by a total of i buyers (including the n-th buyer who visits her with probability 1). Hence, the summation over all possible events yields: $1 + (n-1)\theta$.

Step 2: Back to the seller's problem, replace the term $\sum_{i=1}^{n} H(i,n,\theta)\, i\, p_i^j$ from the constraint in the seller's problem into her profit function. As in the proofs of Lemmas 1, 2, this step allows us to write seller j's objective only as a function of θ. Denote this objective as $J(\theta)$:[21]

$$\max_{\theta} J(\theta) \equiv \theta \left[\sum_{i=1}^{n} H(i,n,\tilde{\theta}) i\, \tilde{p}_i - c \right]. \tag{a.6}$$

Exploiting the link between H, F in (a.5), one can show that

$$J'(\theta) = \sum_{i=1}^{n} H(i,n,\tilde{\theta})\, i\, \tilde{p}_i - nc - n\frac{\theta}{\tilde{\theta}} \frac{1}{m-1} \sum_{i=1}^{n} H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta}) - 1]\, i\, \tilde{p}_i.$$

Given that $\tilde{p}_i = \tilde{a} + \tilde{b}(i-1)$, one can verify that

$$\begin{aligned} \lim_{\theta \to 0} J'(\theta) &= \tilde{a} - c + \tilde{b}\, \frac{n-1}{m-1}, \\ \lim_{\theta \to 1} J'(\theta) &= \tilde{a} - c - \tilde{b}\, \frac{n-1}{m-1}. \end{aligned}$$

Hence, $\lim_{\theta \to 0} J'(\theta) > \lim_{\theta \to 1} J'(\theta)$, and the first-order condition will be necessary for profit maximization if $\lim_{\theta \to 0} J'(\theta) > 0 > \lim_{\theta \to 1} J'(\theta)$.[22] I now show that if the first-order condition is necessary, then it will also be sufficient.

Step 3: The next step is to study $J''(\theta)$. Take the derivative of $J'(\theta)$ with respect to θ, and exploit equation (a.5). After some algebra, we obtain

$$\begin{aligned} J''(\theta) = &\frac{-2}{m-1} \frac{1}{\tilde{\theta}} \sum_{i=1}^{n} H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta}) - 1]\, i\, \tilde{p}_i \\ &+ \frac{\theta}{\tilde{\theta}} \frac{1}{(m-1)^2} \Big\{ \frac{1}{\tilde{\theta}} \sum_{i=1}^{n} H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta}) - 1]^2\, i\, \tilde{p}_i \\ &\qquad - \frac{1}{\tilde{\theta}} \sum_{i=1}^{n} H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta}) - 1]\, i\, \tilde{p}_i \\ &\qquad + \sum_{i=1}^{n} H(i,n,\tilde{\theta}) \frac{i-n}{(1-\tilde{\theta})^2}\, i\, \tilde{p}_i \Big\}. \end{aligned}$$

or, using the fact that $\tilde{p}_i = \tilde{a} + \tilde{b}(i-1)$,

$$J''(\theta) = \frac{-2}{m-1} \frac{1}{\tilde{\theta}} \sum_{i=1}^{n} H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta}) - 1]\, i\, [(\tilde{a} - \tilde{b}) + \tilde{b} i]$$

[21] In the definition of J, I ignore the constant and positive term n.

[22] If $\lim_{\theta \to 1} J'(\theta) > 0$ ($\lim_{\theta \to 0} J'(\theta) < 0$), the optimization problem of the seller has a corner solution given by $\theta = 1$ ($\theta = 0$).

$$+\frac{\theta}{\tilde{\theta}}\frac{1}{(m-1)^2}\Big\{\frac{1}{\tilde{\theta}}\sum_{i=1}^{n}H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta})-1]^2(\tilde{a}-\tilde{b})i+\frac{1}{\tilde{\theta}}\sum_{i=1}^{n}H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta})-1]^2\tilde{b}\,i^2$$

$$-\frac{1}{\tilde{\theta}}\sum_{i=1}^{n}H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta})-1](\tilde{a}-\tilde{b})\,i-\frac{1}{\tilde{\theta}}\sum_{i=1}^{n}H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta})-1]\tilde{b}i^2$$

$$+\sum_{i=1}^{n}H(i,n,\tilde{\theta})\frac{i-n}{(1-\tilde{\theta})^2}(\tilde{a}-\tilde{b})i+\sum_{i=1}^{n}H(i,n,\tilde{\theta})\frac{i-n}{(1-\tilde{\theta})^2}\tilde{b}i^2\Big\}.$$

The curly bracket in the last expression contains six terms. Denote them by $J_i(\theta)$, $i=1,...,6$.[23] With a series of claims, I will prove that the expression in the curly bracket is equal to zero.

Claim 1: $J_3(\theta)=0$. This follows directly from Facts 1 and 2 (notice that all these facts, including Fact 3, hold for any θ, hence, for $\tilde{\theta}$ as well).

Claim 2: $J_4(\theta)=-\tilde{b}(n-1)$. This follows directly from Fact 3.

Claim 3: $J_1(\theta)+J_5(\theta)=0$. This claim is not trivial, so I will prove it in more detail. Notice that

$$\frac{\partial\{H(i,n,\theta)[F(i,n,\theta)-1]\}}{\partial\theta}=\frac{1}{\theta}H(i,n,\theta)[F(i,n,\theta)-1]^2+H(i,n,\theta)\frac{i-n}{(1-\theta)^2}. \quad \text{(a.7)}$$

For future reference, notice that the first term on the right-hand side of (a.7) appears in J_1 and J_2, while the second term appears in J_5 and J_6. This will be a key observation for the proof. Exploiting (a.7), we can write

$$\sum_{i=1}^{n}\Big\{\frac{1}{\theta}H(i,n,\theta)[F(i,n,\theta)-1]^2+H(i,n,\theta)\frac{i-n}{(1-\theta)^2}\Big\}i=$$

$$=\sum_{i=1}^{n}\frac{\partial\{H(i,n,\theta)[F(i,n,\theta)-1]i\}}{\partial\theta}=\frac{\partial\Big\{\sum_{i=1}^{n}H(i,n,\theta)[F(i,n,\theta)-1]i\Big\}}{\partial\theta}=0,$$

where the last equality follows from the fact that $\sum_{i=1}^{n}H(i,n,\theta)[F(i,n,\theta)-1]i=0$ (Facts 1 and 2). Now, it follows directly that $J_1(\theta)+J_5(\theta)=0$.

Claim 4: $J_2(\theta)+J_6(\theta)=\tilde{b}(n-1)$. Again, start from equation (a.7), but this time multliply both sides with i^2 instead of i. We obtain

$$\sum_{i=1}^{n}\Big\{\frac{1}{\theta}H(i,n,\theta)[F(i,n,\theta)-1]^2+H(i,n,\theta)\frac{i-n}{(1-\theta)^2}\Big\}i^2=$$

$$=\sum_{i=1}^{n}\frac{\partial\{H(i,n,\theta)[F(i,n,\theta)-1]i^2\}}{\partial\theta}=\frac{\partial\Big\{\sum_{i=1}^{n}H(i,n,\theta)[F(i,n,\theta)-1]i^2\Big\}}{\partial\theta}=n-1,$$

where the last equality follows from the fact that $\sum_{i=1}^{n}H(i,n,\theta)[F(i,n,\theta)-1]\,i^2=(n-1)\theta$ (Fact 3). It is now easy to see that $J_2(\theta)+J_6(\theta)=\tilde{b}(n-1)$.

[23] For instance, $J_4(\theta)=-\frac{1}{\tilde{\theta}}\sum_{i=1}^{n}H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta})-1]\tilde{b}i^2$.

Summarizing the four claims, we conclude that

$$J''(\theta) = \frac{-2}{m-1}\frac{1}{\tilde{\theta}}\sum_{i=1}^{n} H(i,n,\tilde{\theta})[F(i,n,\tilde{\theta})-1]\, i\,[(\tilde{a}-\tilde{b})+\tilde{b}i].$$

Exploiting Facts 1,2, and 3 one last time, one can write

$$J''(\theta) = \frac{-2\tilde{b}(n-1)}{m-1} < 0.$$

Hence, the first-order condition will be necessary and sufficient.

Step 4: Having established that J is concave, one can now impose symmetry on the first-order condition: $a^j = \tilde{a} = a^*$, $b^j = \tilde{b} = b^*$, and $\theta = \tilde{\theta} = \theta^* = 1/m$. After some algebra, one arrives at the expression in (15). The inequality described in (16) guarantees that ex-post rationality of buyers is not violated. It is important to notice that the suggested solution is consistent with the requirements that guarantee $\lim_{\theta\to 0} J'(\theta) > 0 > \lim_{\theta\to 1} J'(\theta)$ in the typical seller's problem. □

Results used in the proof of Proposition 3 in the text. In the proof of Proposition 3, I use the following two results.

Result 1: Given some (symmetric) equilibrium prices indexed by the pair (a^*, b^*), the equilibrium profit is given by

$$\pi^*(n,m;\mathbf{p}^*) = \frac{n}{m}\left[a^* - c + b^*\frac{n-1}{m}\right].$$

From equation (14), we have

$$\begin{aligned}\pi^*(n,m;\mathbf{p}^*) &= \frac{n}{m}\left\{\sum_{i=1}^{n} H(i,n,\theta)\, i\,[(a^*-b^*)+b^*i] - c\right\} = \\ &= \frac{n}{m}\left\{a^* - b^* + \sum_{i=1}^{n} H(i,n,\theta)\, b^* i^2 - c\right\}, \qquad \text{(a.8)}\end{aligned}$$

where the last equality follows from Fact 1. Moreover, in the proof of Lemma 3 we saw that $\sum_{i=1}^{n} H(i,n,\theta)\, i^2 = 1+(n-1)\theta$ (see footnote 20). Hence, Result 1 follows immediately.

Result 2: The term $\bar{b}$ represents the maximum value of b^* that is consistent with equilibrium. To arrive at the formula for $\bar{b}$ provided in the text, one only needs to set the right-hand sides of (15) and (16) equal to each other and solve with respect to b^*. □

References

Acemoglu, D., and R. Shimer (1999): "Efficient Unemployment Insurance," *Journal of Political Economy*, 107(5), 893–928.

Burdett, K., S. Shi, and R. Wright (2001): "Pricing and Matching with Frictions," *Journal of Political Economy*, 109(5), 1060–1085.

Coles, M. G., and J. Eeckhout (2003): "Indeterminacy and directed search," *Journal of Economic Theory*, 111(2), 265–276.

Cournot, A. (1838): *Recherches sur les principes mathématiques de la théorie des richesses/par Augustin Cournot*. L. Hachette.

Dudey, M. (1992): "Dynamic Edgeworth-Bertrand competition," *The Quarterly Journal of Economics*, 107(4), 1461–1477.

Edgeworth, F. (1897): "La teoria pura del monopolio," *Giornale degli economisti*, 40, 13–31.

Eeckhout, J., and P. Kircher (2010): "Sorting versus screening: Search frictions and competing mechanisms," *Journal of Economic Theory*, 145(4), 1354–1385.

Galenianos, M., and P. Kircher (2009): "Directed search with multiple job applications," *Journal of Economic Theory*, 144(2), 445–471.

Geromichalos, A. (2012): "Directed search and optimal production," .

Hawkins, W. B. (2013): "COMPETITIVE SEARCH, EFFICIENCY, AND MULTI-WORKER FIRMS*," *International Economic Review*, 54(1), 219–251.

Jacquet, N., and S. Tan (2012): "Wage-vacancy contracts and coordination frictions," *Journal of Economic Theory*.

Julien, B., J. Kennes, and I. King (2000): "Bidding for labor," *Review of Economic Dynamics*, 3(4), 619–649.

Kreps, D., and J. Scheinkman (1983): "Quantity precommitment and Bertrand competition yield Cournot outcomes," *The Bell Journal of Economics*, pp. 326–337.

Lang, K. (1991): "Persistent wage dispersion and involuntary unemployment," *The Quarterly Journal of Economics*, 106(1), 181–202.

Lester, B. (2010): "Directed search with multi-vacancy firms," *Journal of Economic Theory*.

——— (2011): "Information and prices with capacity constraints," *The American Economic Review*, 101(4), 1591–1600.

Montgomery, J. D. (1991): "Equilibrium Wage Dispersion and Interindustry Wage Differentials," *The Quarterly Journal of Economics*, 106(1), 163–79.

Peters, M. (1984): "Bertrand equilibrium with capacity constraints and restricted mobility," *Econometrica: Journal of the Econometric Society*, pp. 1117–1127.

Shaked, A., and J. Sutton (1982): "Relaxing price competition through product differentiation," *The Review of Economic Studies*, pp. 3–13.

Tan, S. (2010): "Directed Search and Firm Size," *National University of Singapore mimeo.*

Virág, G. (2011): "High profit equilibria in directed search models," *Games and Economic Behavior*, 71(1), 224–234.

Watanabe, M. (2010): "Middlemen: A Directed Search Equilibrium Approach," *Universidad Carlos III de Madrid mimeo.*