Directed Search and Optimal Production *

Athanasios Geromichalos[†]
Department of Economics
University of California, Davis

December 23, 2011

Abstract

I consider a model of directed search in which strategic sellers advertise general trading mechanisms. A mechanism determines the number of buyers that will get served and the side payments, as a function of ex post realized demand. After observing these advertisements buyers simultaneously visit exactly one seller. Each buyer's expected utility depends on the visiting decisions of other buyers. This dependence becomes especially interesting since the buyers cannot coordinate their visiting strategies. Despite the oligopolistic nature of the model, all symmetric equilibria are constrained efficient. Small markets are characterized by multiple equilibria that are not payoff equivalent, but this indeterminacy vanishes as the size of the market grows. I provide closed form solutions for the matching function under any parameter values, and I show that the production decisions of sellers have a substantial effect on the efficiency of the matching technology.

JEL Classification: C78; E24; J6

Keywords: Directed Search; Efficiency; Multiplicity of Equilibrium; Endogenous Matching Function

* I am grateful to Philipp Kircher and Randall Wright for their continuous guidance and assistance. I would also like to thank Kenneth Burdett, Jan Eeckhout, Manolis Galenianos, and Andrew Postlewaite for useful comments and suggestions. All errors are solely my responsibility.

[†]Contact: ageromich@ucdavis.edu

1 Introduction

In many markets sellers face a stochastic demand, and buyers visit sellers without knowing their total number of customers. If the sellers face capacity constraints, they might not be able to serve all customers who visit their store (if too many show up). On the other hand, if the nature of the good is such that a minimum number of customers is required for the service to be profitable, sellers might not be willing to serve the customers that visit their store (if too few show up). In both cases, if buyers must pay a cost in order to visit more than one sellers, some buyers might not get served precisely due to the inability to coordinate their visiting strategies. I refer to this phenomenon as *frictions*.[1]

To illustrate this economic problem through a concrete example, consider the market for boat trips to the Greek islands during the low season. Ferries operated by different companies depart every morning to the same destinations. Suppose that a trip is profitable only if x or more passengers travel. Since demand is low, the probability that some ferries will get visited by fewer than x customers is high. Often ferry owners have a policy (known to the customers) of canceling the trip in that event.[2] If there are more than x buyers that want to travel on a certain day, they would like to coordinate their visiting decisions so that they all travel. Clearly, this type of coordination is difficult.

A question that arises is whether it is optimal to announce in advance the cancelation of a trip. Ex post, if fewer than x customers visit a certain location, it is obvious that the owner would like to dock the boat. However, these announcements are made ex ante. If all owners follow such a strategy, one of them might have an incentive to deviate and advertise that her company never cancels trips. Such a policy might induce passengers to visit the deviant owner's ferry in order to secure their trip. This could, in turn, lead to a situation in which all sellers advertise that they never cancel trips, which is inefficient. [3]

The ferry example is one of many markets in which the inability of buyers to coordinate their visiting strategies, combined with the sellers' production decisions (to ration some customers, if a certain number of them shows up), can lead to equilibria with frictions. The directed search literature (see for example Montgomery (1991) and Burdett, Shi, and Wright (2001), henceforth BSW) provides a natural framework for studying competition in markets that do not perfectly clear. However, in order to keep the analysis simple, most of these papers place severe restrictions on sellers' production announcements.[4]

[1] If buyers can visit any number of sellers at no cost, the lack of coordination is irrelevant, since buyers who do not get served at the store they visit can keep "sampling" stores. In this setting, the total number of sales (matches) in the economy is equal to the short side of the market. Hence, the definition of market frictions employed here follows Lagos (2000).

[2] A crucial feature of this market is that once a passenger visits a ferry and is informed that the trip is canceled, it is too late to take another ferry. In cases like this, the passengers can get their money back or travel the next day with the same ticket.

[3] This situation resembles the standard Prisoner's Dilemma, in which each player's fear that the other player will not coordinate, leads to an inefficient equilibrium outcome.

[4] Most commonly, they assume that sellers have only one unit for sale. Some exceptions are BSW, Hawkins (2006), Lester (2010), and Tan (2010). As I explain below, my model has important differences compared to each of these papers. Watanabe (2010) considers a model where middlemen hold a bigger inventory than sellers and studies the effect of the size of inventories on the equilibrium bid-ask spread.

Assuming that sellers have a fixed number of units for sale is not only unrealistic for most markets, but has at least two more major drawbacks for the analysis. First, it deprives sellers of an important competition tool: in directed search models, buyers care about the probability of getting served and the price that they will pay if they get served. Hence, sellers might wish to use their production announcements strategically, in order to promise buyers a higher probability of service. The second disadvantage of adopting a fixed-supply assumption, is that it can lead to a misspecified matching function. BSW and Lester (2010) highlight that the model in which sellers can sell two units generates a very different matching function than the model in which sellers can only trade with one unit. Hence, production decisions of sellers affect the efficiency of the matching technology and should be studied more carefully.

In this paper, I provide a general framework that aims to analyze the efficiency properties, the production decisions, and the price determination in markets with the following characteristics. A few strategic sellers face a stochastic demand, which they can affect through a public advertisement. Buyers visit only one seller, and they cannot coordinate their visiting strategies. Hence, if sellers' production announcements involve rationing buyers in some states (which will be determined endogenously), equilibrium will be characterized by frictions. A unique feature of my paper is that I explicitly model consumption externalities that buyers might exert on one-another. Thus, even in the extreme scenario of no frictions (if sellers commit to serve any number of visiting customers), a buyer's expected utility depends on the visiting decisions of others.

Sellers compete with each other for customers by advertising general trading mechanisms. A mechanism determines, as a function of ex post realized demand, the number of buyers that get served and the side payments. Since buyers care about net expected utility, sellers can attract more customers by advertising low prices, a high probability of service, and/or a high utility of consumption conditional on being served. Hence, sellers face the following trade off: ex post, they wish to maximize profit, but ex ante, they promise a high buyer's surplus in order to attract a large number of visitors.

In the baseline model, the number of sellers is fixed and welfare depends only on the production decisions of sellers (different equilibrium prices lead to different sharing rules of the same surplus). Since each seller chooses her production announcements strategically in order to direct customers to her store, and away from rival stores, one might expect inefficient outcomes to arise. I show that constrained efficiency is always achieved in symmetric equilibrium.[5] Hence, the model predicts that ferry owners will indeed announce the cancelation of a trip, if too few customers show up. In a variation of the baseline model, I consider free entry and endogenous determination of the number of sellers, and I find that efficiency can be still achieved in equilibrium.

In related literature, Hawkins (2006) provides an efficiency result in a competitive search environment (in the spirit of Shimer (1996) and Moen (1997)). In that setting,

Godenhielm and Kultti (2010) consider a model with stochastic aggregate demand, in which sellers choose their capacity before posting prices. Some of these papers focus on labor markets, where "firms" open vacancies in order to attract "workers". I adopt the "seller" and "buyer" terminology for generality.

[5] Efficiency is constrained by the lack of coordination among buyers. In other words, the only way to improve upon the equilibrium allocation is to assign buyers to specific sellers.

sellers take as given that they must offer customers a certain level of expected surplus.[6] Hence, in his paper, the element of strategic interaction is absent. The current paper shows that efficiency does not result from the price-taking behavior of sellers. Prescott (1975) considers a model of the market for hotel rooms, in which sellers with local monopoly power face a stochastic demand. He shows that the market will provide the efficient amount of rooms. In that paper, buyers who do not get served at some seller can visit other sellers at no cost. My model provides an analogous result for markets with frictions.

Furthermore, I fully characterize equilibrium prices and show that, in small markets, there exists a continuum of equilibria that are not payoff equivalent (but are all constrained efficient). This indeterminacy result is in the same spirit as Coles and Eeckhout (2003), henceforth CE. In my model, the set of equilibria is richer than in CE, a result of considering a broader class of trading mechanisms. Despite the generality of mechanisms available to sellers, common practices, such as posting a fixed price or an auction, can describe equilibrium behavior.[7] In contrast to findings in CE, sellers' optimal equilibrium need not involve price schemes that are increasing in the number of buyers. Finally, I show that, as the size of the market increases, the sharing rule of the surplus is uniquely pinned down and, hence, indeterminacy vanishes.

An important aspect of my model concerns the matching technology. Directed search models have gained a lot of popularity in recent years, in part because they deliver the matching function endogenously.[8] However, as I explained above, most papers assume that sellers have a fixed supply. My paper goes one step further in the direction of endogenizing the matching function, since I also endogenize the sellers' production announcements, which affect the matching technology substantially, as pointed out by BSW and Lester (2010). But unlike these papers, which also limit the supply of sellers to no more than two units, in my model, sellers are free to serve all the buyers in the market, if they think that this is profitable.

Despite the generality of the mechanisms considered, the model is very tractable. For small markets, I provide a closed form solution for the matching function for any set of parameters. Unlike the BSW framework, the matching technology need not exhibit decreasing returns to scale (DRS). In the large market case, closed form solutions are provided for the buyers' and sellers' arrival rates. The matching technology exhibits approximately constant returns to scale (CRS). The model is well suited to examine the effect of changes in aggregate supply on the number of successful matches along the intensive and extensive margin. Keeping the number of units per buyer fixed, the probability with which buyers get served grows as one moves to economies with less sellers (and bigger average production). Hence, the number of successful matches is more responsive along

[6] This is known as the market utility assumption. See also Montgomery (1991), Lang (1991), Acemoglu and Shimer (1999a), Acemoglu and Shimer (1999b), and Galenianos and Kircher (2009).

[7] There is large literature on competing auctions where buyers have private independent values. This includes McAfee (1993), Peters (1995), Burguet and Sakovics (1999), and Julien, Kennes, and King (2000). Also, Epstein and Peters (1996) and Martimort and Stole (2001) provide revelation principle related results for competing mechanisms.

[8] Lagos (2000) and Shimer (2007) are two prominent examples of papers that highlight the importance of endogenous determination of the matching function.

the intensive margin.

The rest of this paper is organized as follows. In Section 2, I present the basic model and define equilibrium. Section 3 examines the benchmark case of two buyers and two sellers. Section 4 generalizes the analysis for n buyers and m sellers and provides a closed form solution for the matching function. In this section, I also consider endogenous entry of sellers. Section 5 focuses on the case of large markets. I briefly conclude in Section 6.

2 The Model

There are n buyers and m sellers in the market, $n, m \geq 2$. All agents are risk neutral. All buyers are identical and anonymous, and each wishes to purchase one unit of an indivisible good. Each seller can produce $x \leq \xi$ units of the good at cost $c(x)$. Unless otherwise specified, $\xi = n$, i.e. a seller can potentially accommodate every buyer in the market. There is no fixed cost, but this assumption can be relaxed. Buyers' utility from consuming the good depends on the number of customers who get served at a certain location: if a seller serves $x \leq n$ customers, the utility enjoyed by each customer is $u(x)$, with $u(x) \geq 0,\ \forall x \leq n$. I assume that for any $x' > x$, $c(x') \geq c(x)$. No further restrictions on $u(x), c(x)$ are necessary at this stage.

The exchange process consists of two stages. At the first stage, each seller posts an advertisement, which describes the trading mechanism that will be followed at her store, taking as given the mechanisms of her $m-1$ competitors. A mechanism describes at every contingency, i.e. as a function of ex post realized demand, the probability of getting served and the side payments, which depend on whether a buyer gets served or not. Sellers are allowed to post any direct mechanism that treats the buyers symmetrically.[9] At the second stage, buyers observe all the advertisements and choose a probability of visiting each seller, taking as given the strategies of other buyers. Once buyers show up at their preferred location, trade takes place according to the publicly advertised mechanisms. I assume that sellers commit to their advertisements.

Since buyers are anonymous and identical, a seller's mechanism is fully characterized by a pair of vectors which describe prices and production at every possible contingency. A trading mechanism for seller j is defined as $M^j \equiv \{\mathbf{p}^j, \mathbf{k}^j\}$, where $\mathbf{p}^j = (\{p_x^j\}_{x=1}^n, e^j)$ and $\mathbf{k}^j = \{k_x^j\}_{x=1}^n$. For all $x = 1, 2, ..., n$, $k_x^j \leq x$ determines the number of buyers that will get served if x of them visit seller j. Similarly, p_x^j is the price paid to seller j by customers who get served, given that this seller gets visited by x buyers. The term e^j denotes the entry fee paid by all visiting customers.[10] I refer to $\mathbf{p}^j$ as the price scheme/schedule and

[9] A mechanism is indirect if it allows a seller to condition her advertisement on the advertisements posted by other sellers. For a discussion on this class of mechanisms see CE.

[10] Introducing this entry fee allows me to consider the most broad class of symmetric and anonymous direct mechanisms. This needs to be clarified. When a buyer visits a seller, she can be one of the $x \leq n$ buyers who show up, and she can be one of the k_x^j buyers who get served or one of the $x - k_x^j$ buyers who get rationed. If sellers did not have the entry fee tool in their disposal, then buyers who get rationed would always receive a payoff of zero. Allowing sellers to charge a -possibly negative- entry fee, leads to the most general class of mechanisms, in the sense that the promised payoff to a buyer under *any possible*

to $\mathbf{k}^j$ as the production plan/schedule posted by seller j.

In order to have buyers participate in the trading process, the expected utility generated by the posted mechanisms has to be non-negative. Moreover, I assume that buyers can walk away from the trading process at any time and obtain utility equal to zero (due to frictions they cannot visit another seller). Hence, some additional (ex-post) participation constraints for the buyers have to be imposed. These require $p_x^j \leq u(k_x^j)$, i.e. I assume that the utility of a buyer who gets served at seller j depends on that seller's realized production. There is no other restriction on the price schedule a seller can advertise. Prices and the entry fee can be negative. A critical assumption adopted here, is that the entry fee is paid by all buyers who visit a certain location before they find out the total number of visiting customers and whether they will get served or not.[11]

The production plan $\mathbf{k}^j$ chosen by seller j satisfies $\mathbf{k}^j \in \prod_{x=1}^n K_x$, with $K_x = \{0, ..., x\}$ for all $x \geq 1$. I consider mechanisms that lead to non-negative expected profit. If a buyer visits seller j, who advertises $M^j = \{\mathbf{p}^j, \mathbf{k}^j\}$ and gets visited by x customers, she obtains a payoff equal to $u(k_x^j) - p_x^j - e^j$, if she gets served and $-e^j$ otherwise. The seller's profit is given by $k_x^j p_x^j + xe^j - c(k_x^j)$. To describe ex ante payoffs, suppose that a seller who advertises $M = \{\mathbf{p}, \mathbf{k}\}$, gets visited by an arbitrary buyer with probability θ. The expected utility of a buyer who visits that seller is given by[12]

$$U(\theta, M) = \sum_{i=1}^{n} \binom{n-1}{i-1} (1-\theta)^{n-i} \theta^{\,i-1} \frac{k_i}{i} \left[u(k_i) - p_i \right] - e, \tag{1}$$

and the expected profit of that seller is given by

$$\pi(\theta, M) = \sum_{i=1}^{n} \binom{n}{i} (1-\theta)^{n-i} \theta^{\,i} \left[k_i \, p_i + ie - c(k_i) \right], \tag{2}$$

where $\binom{n}{i}$ denotes the binomial coefficient. For future reference it will be useful to define

$$H(n, \theta, i) \equiv \binom{n-1}{i-1} (1-\theta)^{n-i} \, \theta^{\,i-1} \frac{1}{i}. \tag{3}$$

This is the probability with which a buyer gets served when a total of i customers show up at her preferred location, and the seller has one unit for sale.

---

scenario, can be any amount that does not violate her participation constraints (explained below). Thus, there is loss of generality in considering mechanisms without the entry fee.

[11] This assumption is also adopted in the spirit of maintaining maximum generality of the mechanisms available to sellers. If buyers knew the total number of visiting customers before they paid the entry fee, they would prefer to walk away in some states. Then, one would have to restrict attention to mechanisms that yield non-negative utility in every state.

[12] Although this expression is linear in prices, this is not necessarily true regarding the term $u(k_i)$. In other words, buyers care about the distribution of $u(k_i)$, not just the expected value. Hence, when I say that buyers are risk neutral, I mean they are risk neutral with respect to prices. I would like to thank an anonymous referee for pointing out this interesting interpretation.

As it is common in the directed search literature, I focus on symmetric equilibria in which buyers play mixed strategies. [13]

Definition 1. A subgame perfect equilibrium is a collection of mechanisms M^j, $j = 1, ..., m$ and a strategy $s : \prod_{j=1}^{m} M^j \longrightarrow \Delta_m$, such that:

i) Given the posted mechanisms, the strategies $s^i = s$, $i = 1, ..., n$ maximize buyers' expected utility, and

ii) Given buyers' strategy s, M^j is a best response to the mechanisms announced by other sellers, for all $j = 1, ..., m$.

The term Δ_m denotes the unit simplex that captures the probabilities with which buyers visit each seller.

Definition 2. Let $\sigma(x) \equiv xu(x) - c(x)$ denote the surplus generated if a seller serves x customers. I refer to $\sigma : \mathbb{N} \to \mathbb{R}$ as the ex post surplus function.

3 The 2×2 Case

In this section I analyze the benchmark case with $n = m = 2$, which provides important intuition for the general n, m case. First, I assume that sellers take $\mathbf{k}$ as given and derive equilibrium prices. Then, I allow sellers to choose production endogenously and examine the conditions under which different values of $\mathbf{k}$ survive in equilibrium. Throughout the analysis I focus on symmetric equilibria for the sellers.

3.1 Exogenous k

Let the sellers be labeled A and B. Assume for simplicity that sellers always serve at least one customer. With $n = 2$, we can only have $\mathbf{k} = (1, 1)$ or $\mathbf{k} = (1, 2)$. Assume $\sigma(1), \sigma(2) > 0$. First, suppose $\mathbf{k} = (1, 1)$. Sellers take this as given and choose a price schedule, $\mathbf{p}^j = (p_1^j, p_2^j, e^j)$, $j = A, B$. The following proposition describes the full set of symmetric equilibria.

Proposition 1. *Every triplet* $(p_1, p_2, e) = \left(\frac{u(1)+c(1)}{2} - \epsilon, \alpha, \epsilon\right)$ *that satisfies*

$$\alpha \leq u(1), \epsilon \geq \frac{1}{2}[c(1) - u(1)], \text{ and } \alpha + 2\epsilon \in [2c(1) - u(1), 2u(1) - c(1)], \tag{4}$$

constitutes a symmetric equilibrium in the model with $n = m = 2$ *and* $\boldsymbol{k} = (1, 1)$.

[13] Equilibria in pure strategies still exist, but they are considered implausible since they require an unreasonable degree of coordination among the buyers (i.e. a buyer needs to know where other buyers are going). Equilibria in mixed strategies are more interesting, because they are consistent with the search frictions that directed search models were designed to capture in the first place. For a more detailed discussion on the topic see BSW.

Proof. Let θ be the probability with which an arbitrary buyer visits seller A and U_j the expected utility of a buyer who visits seller j. Using (1), one can write

$$\begin{aligned} U_A &= (1-\theta)[u(1)-p_1^A] + \frac{\theta}{2}[u(1)-p_2^A] - e^A, \\ U_B &= \theta[u(1)-p_1^B] + \frac{1}{2}(1-\theta)[u(1)-p_2^B] - e^B. \end{aligned}$$

Without loss of generality, focus on seller A who wishes to maximize

$$\begin{aligned} \pi_A &= \theta^2[p_2^A + 2e^A - c(1)] + 2\theta(1-\theta)[p_1^A + e^A - c(1)] \\ &= 2\theta\left[\frac{\theta}{2}p_2^A + (1-\theta)p_1^A + e^A - \left(1-\frac{\theta}{2}\right)c(1)\right], \end{aligned}$$

subject to $U_A = U_B$. Notice that the term $(\theta/2)p_2^A + (1-\theta)p_1^A + e^A$ in the profit function, also appears in the expression U_A. Hence, one can conveniently re-write the seller's objective as a function of the variable θ only. In particular, the seller solves

$$\max_{\theta\in[0,1]}\left\{2\theta\left[u(1) - \frac{\theta}{2}u(1) - U_B\right] - \theta(2-\theta)c(1)\right\}.$$

Even though the seller maximizes profits subject to $U_A = U_B$, this does not rule out the possibility of her choosing to attract buyers with probability one.

The first-order condition for an interior maximum implies[14]

$$u(1) - \frac{\theta}{2}u(1) - U_B + \theta\left[-\frac{u(1)}{2} - \frac{\partial U_B}{\partial\theta}\right] - (1-\theta)c(1) = 0, \tag{5}$$

where $\partial U_B/\partial\theta = u(1)/2 - p_1^B + p_2^B/2$. Symmetry requires $p_1^A = p_1^B = p_1$, $p_2^A = p_2^B = p_2$, $e^A = e^B = e$, and $\theta = 1/2$. Imposing these conditions on (5) yields

$$p_1 + e = \frac{1}{2}[u(1) + c(1)].$$

The conditions $\alpha \le u(1)$, $\epsilon \ge [c(1) - u(1)]/2$ guarantee ex post participation of the buyer in every possible state. The condition $\alpha + 2\epsilon \in [2c(1) - u(1), 2u(1) - c(1)]$ guarantees that expected utility and profit are non-negative. □

Proposition 1 reveals that equilibrium prices are not uniquely pinned down. There are two degrees of freedom in determining the three equilibrium objects (p_1, p_2, e). The set of possible equilibria is broader than the one identified in CE, a result of allowing sellers to advertise a more general class of mechanisms. Figure 1 illustrates the set of all possible equilibria.[15] These equilibria are not payoff equivalent. Sellers achieve maximum

---

[14] This condition will also be sufficient if $c(1) - 2u(1) + 2p_1^B - p_2^B < 0$. I focus on cases where this condition holds. If not, seller A sets $\theta = 1$ and a symmetric equilibrium cannot be supported.

[15] The assumption, according to which buyers pay the entry fee before they find out the total number of visitors, is essential for the existence of some equilibria. For example, consider the equilibrium price scheme $(p_1, p_2, e) = (c(1), u(1), \sigma(1)/2)$. If a seller gets visited by two buyers, and the buyers know this before they pay the entry fee, they prefer to walk away and obtain zero utility, rather than pay a positive fee in order to enter the store and play a mechanism that yields zero with certainty.

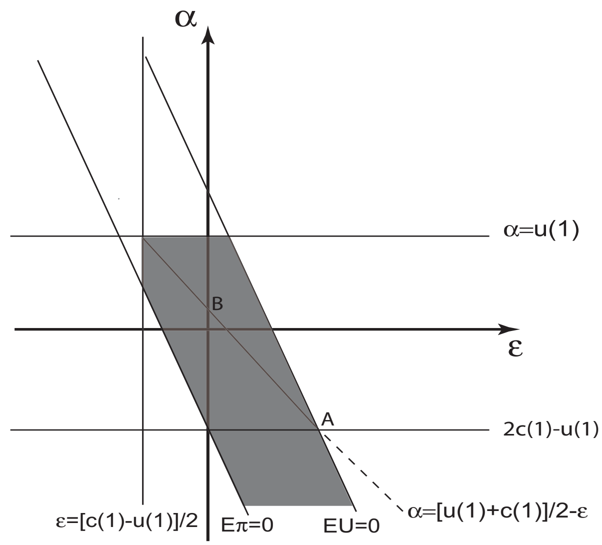

Figure 1: All α, ϵ in the shaded region are consistent with equilibrium.

expected profit when $\alpha + 2\epsilon = 2u(1) - c(1)$ (the line labeled as $EU = 0$ in Figure 1). On the other extreme, there exist equilibria in which all the surplus goes to the buyers (the line labeled as $E\pi = 0$). This happens when $\alpha + 2\epsilon = 2c(1) - u(1)$. In section 4, I show that the indeterminacy result holds true for any finite n, m, and I provide some intuition.

In this paper, I do not provide a theory of refining equilibria. However, a few interesting cases are worth emphasizing on. First, the case in which $\alpha = u(1)$ replicates the outcome of an auction among identical buyers. In CE this is the unique optimal equilibrium for the sellers. Here, the auction equilibrium is preferred by sellers only if $\epsilon = \sigma(1)/2$. Competition in fixed prices also describes equilibrium behavior. Every (α, ϵ), with $\alpha = [u(1) + c(1)]/2 - \epsilon$ and $\epsilon \in [-\sigma(1)/2, 3\,\sigma(1)/2]$ is a symmetric equilibrium in which $p_1 = p_2$. There exists an equilibrium in fixed prices, in which sellers extract all the surplus. It is indicated by point A in Figure 1. Point B represents the equilibrium studied in BSW, i.e. an equilibrium with a fixed price schedule and no entry fee.

Next, consider the case in which $\mathbf{k} = (1, 2)$. Proposition 2 describes the set of symmetric equilibria for this case.

Proposition 2. *Every triplet* $(p_1, p_2, e) = \left(u(1) - u(2) + \frac{c(2)}{2} - \epsilon, \alpha, \epsilon\right)$ *that satisfies*

$$\alpha \leq u(2), \epsilon \geq \frac{1}{2}c(2) - u(2), \text{ and } \alpha + \epsilon \in \left[u(2) - \sigma(1), 2u(2) - \frac{1}{2}c(2)\right],$$

constitutes a symmetric equilibrium in the model with $n = m = 2$ *and* $\boldsymbol{k} = (1, 2)$.

Proof. The proof is similar to that of Proposition 1. Therefore, it is omitted. □

The equilibrium price p_1 is negatively related to $u(2)$. Intuitively, the willingness to pay a little more in order to be the only customer at a specific store, increases when $u(2)$ falls. Once again, a continuum of equilibria exist, that are not payoff equivalent. Sellers achieve maximum profit when $\alpha+\epsilon = 2u(2)-c(2)/2$. Although $\alpha = u(2)$ is part of an equilibrium, it does not replicate the outcome of an auction. Since there is no rationing, buyers do not have an incentive to bid against each other for the good. Finally, competition in fixed prices does not always describe equilibrium behavior. Sufficient conditions for this type of equilibrium to exist, are $\sigma(2)/2 \leq 2u(1) - c(1)$ and $3u(2) - c(2) \geq u(1)$.

3.2 Endogenous Determination of k

Suppose that sellers choose $\mathbf{k}$ together with a price schedule. Proposition 3 establishes conditions under which different values of $\mathbf{k}$ emerge in equilibrium.

Proposition 3. *In the model with $n = m = 2$, consider the game in which sellers post general mechanisms $M^j = \{\boldsymbol{p}^j, \boldsymbol{k}^j\}$.*

a) The production plan $\boldsymbol{k} = (1,1)$ is posted in the symmetric equilibrium if and only if $\sigma(1) \geq \sigma(2)$. Equilibrium prices are described in Proposition 1.

b) The production plan $\boldsymbol{k} = (1,2)$ is posted in the symmetric equilibrium if and only if $\sigma(1) \leq \sigma(2)$. Equilibrium prices are described in Proposition 2.

Proof. a) To see why the suggested strategies constitute an equilibrium, fix the strategy of seller B to $\mathbf{k}^B = (1,1)$ and $\mathbf{p}^B = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$, where (α, ϵ) satisfy (4). The announcement $\mathbf{k} = (1,1)$ will be posted in equilibrium, if even the best possible deviation for seller A, that involves $\mathbf{k}^d = (1,2)$, is not profitable. To find the best deviation, suppose $\mathbf{k}^A = \mathbf{k}^d = (1,2)$ and $\mathbf{p}^A = \mathbf{p}^d = (p_1^d, p_2^d, e^d)$, and let t denote the probability with which an arbitrary buyer visits seller A. The expected payoffs from visiting sellers A or B are

$$\begin{aligned} U_d &= (1-t)[u(1) - p_1^d] + t[u(2) - p_2^d] - e^d, \\ U &= t\left[\frac{1}{2}u(1) - \frac{1}{2}c(1) + \epsilon\right] + \frac{1}{2}(1-t)[u(1) - \alpha] - \epsilon. \end{aligned} \tag{6}$$

The deviant seller wishes to solve

$$\max_t \left\{2t\left[(1-t)u(1) + tu(2) - U\right] - 2t(1-t)c(1) - t^2c(2)\right\},$$

where U is given by (6). The optimal choice of t is given by

$$t^* = \frac{u(1) + \alpha + 2\epsilon - 2c(1)}{2\left[2u(1) - 2u(2) + \alpha + 2\epsilon - 3c(1) + c(2)\right]},$$

if $\sigma(2) < \sigma(1) + \alpha + 2\epsilon + u(1) - 2c(1)$ and $t^* = 1$, otherwise. Hence, the maximum profit seller A can achieve if she deviates is

$$\pi_A^d = \frac{1}{4}\frac{[u(1) + \alpha + 2\epsilon - 2c(1)]^2}{2u(1) - 2u(2) + \alpha + 2\epsilon - 3c(1) + c(2)},$$

if $\sigma(2) < \sigma(1) + \alpha + 2\epsilon + u(1) - 2c(1)$, and $\pi_A^d = \sigma(2) - \sigma(1)$, otherwise.

On the other hand, if seller A "follows" the prescribed strategy, she obtains

$$\pi_A^f = \frac{1}{4}[u(1) + \alpha + 2\epsilon - 2c(1)].$$

Comparing π_A^f with the adequate expression for π_A^d, implies that seller A has no incentive to deviate (in the sense that even her best possible deviation is not good enough), if and only if $\sigma(1) \geq \sigma(2)$.

b) The proof is similar to that of part (a). Therefore, it is omitted. □

Proposition 3 reveals that equilibrium production advertisements are ex post efficient, in the sense that sellers choose to serve a second buyer if and only if the $\sigma(2) \geq \sigma(1)$. The equilibrium production plan is generically unique. However, associated with every equilibrium $\mathbf{k}$, is a continuum of prices described in Propositions 1 and 2. Therefore, for any parameter values there exist multiple symmetric equilibria that are all ex post efficient but not payoff equivalent.

In the proof of Proposition 2, one might expect seller A to have an incentive to deviate to $\mathbf{k}^A = (1, 2)$: this strategy could be beneficial, since it might attract buyers who do not want to risk getting rationed. Nevertheless, if $\sigma(1) \geq \sigma(2)$, seller A has no profitable deviation that involves $\mathbf{k}^A = (1, 2)$. Intuitively, if $\sigma(1) \geq \sigma(2)$ we either have $u(1) \approx u(2)$ and $c(2) >> c(1)$, or $c(1) \approx c(2)$ but $u(2) << u(1)$. In the first case, setting $\mathbf{k}^A = (1, 2)$ is attractive for buyers, but expensive for sellers. In the second case, the marginal cost of the second unit is insignificant. However, buyers do not value the certainty of getting served, because their utility in the event of a double coincidence at seller A is very low.

Consider the total welfare in the economy. If the symmetric equilibrium announcement is $\mathbf{k}_1 = (1, 1)$, the total expected surplus in the economy is $S(\mathbf{k}_1) = 3\sigma(1)/2$. If $\mathbf{k} = \mathbf{k}_2 = (1, 2)$, we have $S(\mathbf{k}_2) = \sigma(1) + \sigma(2)/2$. Clearly, $S(\mathbf{k}_2) \geq S(\mathbf{k}_1)$, if and only if $\sigma(2) \geq \sigma(1)$, which is the condition under which $\mathbf{k}_2$ is posted in equilibrium. Hence, for any parameter values, the emerging equilibrium is not only ex post efficient, in the sense described above, but also ex ante efficient, in the sense that it involves the realization of $\mathbf{k}$ that maximizes expected total surplus. In the next section I generalize this result for any n, m.

4 The $n \times m$ case

4.1 The Social Planner's Problem

Since one of the main results of this section concerns the efficiency of sellers' production choices, I start by describing the notion of efficiency that I employ. For given n, m, the Social Planner chooses a production plan, to be posted by all sellers, in order to maximize expected total surplus.[16] The following definition describes this plan.

[16] Strictly speaking, the Planner also chooses the probability with which a typical buyer visits each seller. However, we know from BSW that the expected surplus is maximized when buyers visit each seller

Definition 3. A production plan $\mathbf{k} = \{k_i\}_{i=1}^n$ is called ex ante constrained efficient, if it maximizes the expected total surplus,

$$S(\mathbf{k}) = n \sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right) \sigma(k_i), \tag{7}$$

where H is defined in (3).

For practical purposes, it is useful to also introduce another definition of efficiency.

Definition 4. A production plan $\mathbf{k} = \{k_i\}_{i=1}^n$ is called ex post efficient, if for all $i = 1, 2, ..., n$, $k_i = \arg\max_{\{x \leq i\}} \sigma(x)$.

The total expected surplus in (7) is equal to the expected number of matches in the economy, adjusted for the surplus generated when i matches are formed. I refer to a production plan that achieves maximum expected surplus as constrained (rather than unconstrained) efficient, because, as it is standard in directed search models, the Social Planner is limited by the lack of coordination among buyers. In other words, the Planner cannot bypass the coordination frictions by "assigning" various buyers to different sellers. Finally, definition 4 describes an ex post efficient production plan. Such a plan maximizes the total surplus, conditional on the number of buyers that show up at a specific seller. By construction, an ex post efficient plan $\{k_i\}_{i=1}^n$ is non-decreasing in i.[17]

In order to show that a symmetric equilibrium is efficient one needs to establish that the production plan posted by sellers is ex ante efficient. A direct proof of this result is not easy. However, in what follows, I show that all equilibrium production plans are ex post efficient. Then, I conclude that all symmetric equilibria are constrained efficient, by showing that a production plan is ex ante efficient if and only if it is ex post efficient.

4.2 Equilibrium in the General Model

This subsection describes equilibrium and establishes a generalized efficiency result in the model with finite n, m. The methodology followed in the previous section is not useful when n is big. For a general n (and assuming $k_1 = 1$), there are $(n/2)(n+1)$ possible values of $\mathbf{k}$. In order to apply the method developed in Section 3, one would first have

with equal probability, because this minimizes the congestion effects. Showing this result formally in my model would be a trivial extension of the proof in BSW. In what follows, I take as given that the Planner "asks" buyers to visit each seller with probability $1/m$.

[17] A practical way to identify the ex post efficient production is the following: if one buyer shows up it is efficient to serve her *iff* $\sigma(1) > 0$. If two buyers arrive, it is efficient to serve both *iff* $\sigma(2) \geq \sigma(1)$. Suppose that $\sigma(2) < \sigma(1)$, and so efficiency requires rationing the second customer. In order to find what is the efficient thing to do when three buyers show up, one only needs to check whether $\sigma(3) \geq \sigma(1)$. Arguing in a similar fashion, the process of identifying the ex post efficient $\mathbf{k}$, can be viewed as a tournament where, in round i, $\sigma(i)$ is compared to the surplus that "won" in the preceding round. Then, the "winner" proceeds to round $i+1$, in which it is compared to $\sigma(i+1)$. This process goes on until $i = n$.

to establish equilibrium prices associated with each value of $\mathbf{k}$, and then check for the existence of profitable deviations associated with each of the remaining $(n/2)(n+1)-1$ values. This process would have to be repeated $(n/2)(n+1)$ times. Lemma 1 below greatly simplifies the task of characterizing equilibrium, by establishing a useful result that concerns sellers' optimal choice of $\mathbf{k}$.

Let S^i denote seller i's strategy set, and let $\mathbf{k}^*$ be ex post efficient.

Lemma 1. *Fix the strategy of all sellers but j to some arbitrary $M^{-j} \in \prod_{h \neq j} S^h$, and the strategy of all buyers to some arbitrary $(\theta_1, ..., \theta_{m-1}) \in \Delta_m$. For any $M \in S^j$, there exists $\boldsymbol{p}^* = (p_1^*, p_2^*, ..., p_n^*, e^*)$, with $p_i^* \leq u(k_i^*)$ for all i, such that*

$$\pi_j\left(\theta_j, M^*; M^{-j}\right) \geq \pi_j\left(\theta_j, M; M^{-j}\right),$$
$$s.t.\ U_j\left(\theta_j, M\right) = U_j\left(\theta_j, M^*\right) = U_h(\theta_h, M^{-j}),$$

where $M^* = \{\boldsymbol{p}^*, \boldsymbol{k}^*\}$, $\pi_j\left(\theta_j, M; M^{-j}\right)$ is seller j's profit if she advertises M, given M^{-j}, and $U_k(\theta_k, M^k)$ is the expected utility from visiting seller k, when the latter posts M^k and gets visited by an arbitrary buyer with probability θ_k.

Proof. See the appendix. □

Regardless of the strategy followed by other sellers, seller j's best response involves announcing an ex post efficient $\mathbf{k}^*$. More specifically, for a given M^{-j} and any $M \in S^j$, seller j can always find a price scheme $\mathbf{p}^*$, which, together with an ex post efficient $\mathbf{k}^*$, leaves buyers indifferent between visiting seller j or any other seller in the second stage, and improves this seller's profit (strictly if the announcement $\mathbf{k}$ in M is not ex post efficient). Hence, if a symmetric equilibrium exists, it must contain ex post efficient production schedules.[18]

The next proposition builds on Lemma 1 in order to describe equilibrium prices and establish existence of symmetric equilibrium. Define $f(n,m,i) \equiv i - (n-i)/(m-1)$.

Proposition 4. *a) Strategies $M^* = \{\boldsymbol{p}^*, \boldsymbol{k}^*\}$ and $s^* = (1/m, ..., 1/m)$, constitute a symmetric equilibrium, if and only if $\boldsymbol{k}^*$ is ex post efficient and $\boldsymbol{p}^*$ solves*

$$\sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right)\left[1 - \frac{f(n,m,i)-1}{m-1}\right] k_i^* p_i^* + e^* =$$
$$= \sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right)\left\{\frac{m\left[1 - f(n,m,i)\right]}{m-1} k_i^* u(k_i^*) + f(n,m,i)\, c(k_i^*)\right\}, \tag{8}$$

with $p_i^* \leq u(k_i^*),\ \forall i$, and $\pi(M^*), U(M^*) \geq 0$.

b) If $k_i^* \neq 0\ \forall i$, a price $\boldsymbol{p}^*$ that satisfies these conditions always exists. Since $\{k_i^*\}_{i=1}^n$ is non-decreasing in i, a sufficient condition for $k_i^* \neq 0\ \forall i$ to hold, is that $\sigma(1) > 0$.

[18] Adequately modified versions of Lemma 1 would hold in various, more general, settings. One example is an environment with heterogeneous buyers and/or sellers.

Proof. See the appendix. □

There is only one equation that characterizes $\mathbf{p}^*$, which implies that there are n degrees of freedom in the determination of prices and e^*. Moreover, any sharing rule of the surplus can be supported in equilibrium, i.e. $\pi(M^*) \in [0, S(\mathbf{k}^*)/m]$. This follows directly from the fact that e^* enters (8) in a linear fashion. Hence, the indeterminacy of the sharing rule identified in the 2×2 case persists in the general, finite, n, m case. The intuition is as follows. In the finite environment, the utility in the subgame changes when a single seller deviates, that is, the market utility is not constant. If other sellers offer a good deal only if a consumer is alone but not if several consumers show up, then as the deviant seller withdraws customers from other sellers, these other firms become more attractive because any buyer who goes to them has a higher chance of being alone. How quickly the attractiveness of other firms changes depends on the mechanisms that these firms have in their disposal. In other words, when a seller has access to general pricing mechanisms, she can advertise more buyer surplus in some states and less in other states, leaving expected payoffs of other players unchanged. Hence, for any price announcement by her rival sellers, seller j has a best response correspondence (rather than a function), which gives rise to a continuum of equilibria that are not payoff equivalent.

As the discussion above reveals, the rich set of equilibria that arises is a result of the generality of the mechanisms available to the sellers. If sellers compete in fixed prices, like in BSW, the sharing rule of the surplus will be uniquely pinned down. Moreover, fixed prices or auctions do not always constitute equilibrium. The proof of existence of symmetric equilibrium (part (b) of Proposition 4) exploits the fact that sellers can charge different prices under different realizations of ex post demand. Hence, if $p_i^* = p^*$ for all i, it is not guaranteed that the proof will go through.

As in the previous section, here I do not provide a theory of refining the various equilibria. However, in the next section I show that as the market grows large, the equilibrium sharing rule is uniquely pinned down. In what follows, I present some examples that help illustrate the importance of Proposition 4 and highlight some interesting insights. The first is an artificial numerical example that shows how the proposition can be used in order to generate the set of all possible market equilibria. The next two are examples of realistic markets that my model can help analyze.

First, suppose $n = 4$, $m = 2$, $\{u(i)\}_{i=1}^4 = \{1, 1, 1, 1\}$, and $\{c(i)\}_{i=1}^4 = \{0, 0.5, 1.6, 3\}$. Since here $\{\sigma(i)\}_{i=1}^4 = \{1, 1.5, 1.4, 1\}$, the ex post efficient production is $\mathbf{k}^* = (1, 2, 2, 2)$. Then, (8) implies that every $(p_1^*, p_2^*, p_3^*, p_4^*, e^*)$ that satisfies $4p_1^* + 6p_2^* - p_4^* + 8e^* = 15/2$, $p_1^* + 3p_2^* + 2p_3^* + p_4^*/2 + 8e^* \in [11/8, 13]$, and $p_i^* \leq 1$ for all i, is an equilibrium price scheme. Setting $p_i^* = p^*$ for all i and $e^* = 0$, yields $p^* = 5/6$. The price scheme $\mathbf{p}^* = (5/6, 5/6, 5/6, 5/6, 0)$ satisfies ex post rationality for the buyers and leads to positive expected profit and utility. Hence, the class of mechanisms considered in BSW also describes equilibrium behavior here.

A second interesting class of equilibria is the one that replicates an auction. A necessary condition for this type of equilibrium to exist, is some ex post shortage of supply. More formally, if $\mathbf{k}^*$ is ex post efficient, an auction equilibrium exists only if $Z \equiv \{x \in \{1, 2, ..., n\} : k_x^* < x\} \neq \emptyset$. In this case, an auction equilibrium satisfies

$p_i^* = u(k_i^*)$, $\forall i \in Z$. In the numeric example studied above, the auction equilibrium requires $p_3^* = p_4^* = 1$. Every triplet (p_1^*, p_2^*, e^*) that satisfies $4p_1^* + 6p_2^* + 8e^* = 17/2$, and $p_1^* + 3p_2^* + 8e^* \in [-9/8, 21/2]$ is (part of) an auction equilibrium.

A well known result from the related literature (for example CE and Virag (2007)), is that the sellers' optimal equilibrium involves price schemes that are increasing in the number of visiting customers. This is not necessarily true here, as the numeric example considered above illustrates. The vector $(p_1^*, p_2^*, p_3^*, p_4^*, e^*) = (-1/4, -5/12, 1, 1, 3/2)$ is an equilibrium price scheme and leads to zero expected utility for the buyers. However, it is not increasing since $p_1^* > p_2^*$. On the other hand, my model does share the following common finding with Virag (2007): when the sellers cannot charge a positive entry fee, they cannot achieve full extraction of the surplus in equilibrium. This result is described in the following proposition.

Proposition 5. *Suppose sellers cannot charge a positive entry fee. Then, in all symmetric equilibria* $U(M^*) > 0$*.*

Proof. See the appendix. □

Next, I consider the example of boat trips to Greek islands described in the Introduction. The numbers are hypothetical, but the functional forms are chosen in order to replicate the salient features of this market. Let $n = 500$ and $m = 5$. Sellers advertise a production plan $\mathbf{k}$ and a single price p payed by all customers who end up traveling. Each customer's valuation of a trip is $u = 1$. Once a trip takes place, it has a certain fixed cost regardless of the number of passengers that travel, i.e. $c(i) = c = 25$ for all $i \geq 1$, and $c(0) = 0$. Also, assume that each boat has a capacity of 100 travelers.

Here $\sigma(i) = 0$, for $i = 0$ and $\sigma(i) = i - 25$, otherwise. The equilibrium production announcement, which can only be ex post efficient, is the following: $k_i^* = 0$, for all $i \leq 25$, $k_i^* = i$, for all $25 < i \leq 100$, and $k_i^* = 100$, for all $i > 100$.[19] Boat owners never set $k_i^* = i$ for $i \leq 0$, although this might seem tempting ex ante (it could attract customers who wish to secure their trip). This behavior is explained by Lemma 1: sellers can always do better by posting an efficient $\mathbf{k}^*$. Setting $e^* = 0$ and $p_i^* = p^*$ for all i in Proposition 4, implies that $p^* = 0.546$. The profit made by each firm is $\pi^* = 27.693$. Notice that a sufficient condition for existence of symmetric equilibrium is that $k_i^* \neq 0$ for all i. Here $k_i^* = 0$ for some i, however, equilibrium prices satisfying (8) (and participation constraints) exist. In fact, competition in fixed prices, *a la* BSW, describes equilibrium behavior.

The last example I consider illustrates a case of negative consumption externalities. Consider a market with $m = 5$ restaurants and $n = 150$ customers. Customers' utility depends on how crowded their preferred restaurant is (if they get served, if not, $u = 0$). If i represents the number of buyers that end up getting served at a restaurant, the utility

[19] Another example of a company that advertises schemes similar to the boat owners is Groupon, which has gained some publicity recently. Groupon offers "deals of the day", but the offers are valid only if a minimum number of customers subscribes. Despite the obvious similarities, the Groupon example has one big difference. A customer, who is about to decide whether to participate or not, knows how many other customers have already subscribed. Hence, in that example there is room for coordination.

of each buyer is $u(i) = \max\{1 - 0.02(i-1), 0\}$. Restaurants face $c(i) = 0.01i^2$. They advertise a single price, payed by all customers who get served, and a production scheme $\mathbf{k}^*$, which one can think of as the number of tables that the owner chooses to have in the store. More tables, on average, lead to more sales. But the sellers also realize that serving too many people will diminish the customers' valuation of the service. According to Proposition 4, all restaurants will post the ex post efficient schedule. Here, $k_i^* = i$, for $i \leq 17$, and $k_i^* = 17$, for $i > 17$. In words, the restaurants set 17 tables in the store. It can be easily verified that the equilibrium price solves $p^* = 0.680$, and the equilibrium profit of each seller is $\pi^* = 8.669$.

How will the symmetric equilibrium change in the absence of the externalities? Suppose that every buyer who gets served enjoys $u(i) = u = 1$ (this is the utility she enjoys in the pervious example when she is the only customer). With the new utility function, $k_i^* = i$, for $i \leq 25$, and $k_i^* = 25$, for $i > 25$. The new equilibrium price is $p' = 0.606$, and the profit is $\pi' = 8.940$: although restaurants are able to serve more customers, their profit changes only slightly and the price goes down. This might seems paradoxical at first. In the no-externality case, the average valuation of customers for the service is higher (because, typically, customers will not end up eating alone). One might expect that sellers should charge more for a good that buyers value more. But this is only half of the story. There is another important force captured by the model. In the no-externality case, sellers serve up to 25 customers (versus 17). With $n/m = 30$ in this example, customers are confident that their chances of getting served are very high. Since the threat of rationing customers weakens, sellers compete harder on the price end, which restricts them from increasing their profits and, in fact, reduces prices.[20]

I conclude this section with an importnat result that connects the notion of ex post and ex ante efficiency of production schedules. Proposition 4 characterizes symmetric equilibrium and establishes its existence. Also, it guarantees that all equilibrium production plans will be ex post efficient. The notion of ex post efficiency is very useful for the characterization of equilibria, because it provides a simple method of identifying the equilibrium $\mathbf{k}^*$. However, the natural measure of welfare in this model is (ex ante) expected surplus, and equilibrium will be efficient only if the posted $\mathbf{k}^*$ maximizes this measure. This will, in turn, be true only if ex post and ex ante constraint efficiency coincide. The following Lemma dictates that this is indeed the case.

Lemma 2. *A production plan $\boldsymbol{k}^*$ is ex post efficient if and only if it is ex ante constrained efficient.*

Proof. See the Appendix. □

Corollary 1. *Every symmetric equilibrium of the model is constrained efficient, in the precise sense that it involves a production plan that maximizes the total expected surplus.*

[20] A similar point is made in Section 5.2, where I study the case of a large market. Moreover, as the analysis above suggests, this result relies on the specific value of n/m. If n increases to 200, one can show that $\pi^* = 8.670$ (i.e. it hardly changes), but $\pi' = 15.916$.

4.3 The Matching Function

In symmetric equilibrium, the number of expected sales in the economy is given by

$$M(n,m;\mathbf{k}^*) = m\,L(n,m;\mathbf{k}^*) = m\sum_{i=1}^{n} H\left(n,\frac{1}{m},i\right)k_i^*, \tag{9}$$

where $L(n,m;\mathbf{k}^*)$ is the expected number of sales per seller, given the efficient plan $\mathbf{k}^*$. $M(n,m;\mathbf{k}^*)$ can be thought of as the matching function. It depends not only on the number of buyers and sellers in the market, as it is common in the search literature, but also on the production decisions of sellers.

In order to study the efficiency of the matching function, define $A_B \equiv M(n,m;\mathbf{k}^*)/n$, the probability with which a buyer gets served, and $A_S \equiv M(n,m;\mathbf{k}^*)/m$, the average expected sales, in the symmetric equilibrium where $\mathbf{k}^*$ is posted. Also, define market tightness, $b \equiv n/m$. In BSW, where $\mathbf{k}^* = (1,1,...,1)$ by assumption, it is shown that, for n,m finite, these rates decrease as n,m grow and b is held fixed. Hence, the matching technology exhibits DRS. Here, this need not be true, as the following counter-example demonstrates. Let $u(i) = 1\ \forall i$, $c(i) = 1+\epsilon$, $\epsilon \in (0,1)$, for all $i \geq 1$, and $c(0) = 0$, implying $\mathbf{k}^* = (1,1,3,4,...)$. If $n = m = 2$, then $A_S = 0.75$. But if the number of market participants is doubled ($m = n = 4$), A_S increases to 0.79. Hence, the matching technology exhibits increasing returns (around $m = n = 2$).

The result above depends crucially on the choice of parameters, which leads to a quite irregular $\mathbf{k}^*$ (in the sense that $k_1^* = k_2^* = 1$, but then k_3^* "jumps" to 3). If one focuses on more well-behaved production plans, the matching technology typically exhibits DRS in the finite market. One example is a class of schedules that satisfy $k_i^* = i$, for $i \leq \lambda$ and $k_i^* = \lambda$, for $i > \lambda$, for some $\lambda \in \{1,...,n\}$. In words, each seller has λ units for sale, and the "store is open until the good is sold". Such a $\mathbf{k}^*$ could arise if c is convex, and the function $iu(i)$ is concave, one of them strictly.[21] Restricting attention to this case, one can now substitute for $\mathbf{k}^*$ in (9) and, after some manipulations, obtain

$$M(n,m;\mathbf{k}^*) = n\left[1-\sum_{i=\lambda+1}^{n} H\left(n,\frac{1}{m},i\right)(i-\lambda)\right].$$

Also, the probability with which a buyer gets served is

$$A_B = 1-\sum_{i=\lambda+1}^{n} H\left(n,\frac{1}{m},i\right)(i-\lambda). \tag{10}$$

Keeping the market tightness $b = n/m$ fixed, one can show that A_B and $A_S = bA_B$ are decreasing in n. Another interesting fact emerges from inspection of (10). Consider a market with $n = 20$, and a total supply of 12 units. If this supply comes from 12 sellers,

[21] A more formal proof of this result can be found in Proposition 7, in Section 5, where the case of large markets is considered. When $n \to \infty$, $\mathbf{k}^*$ is essentially an infinite sequence. Hence, imposing some structure that keeps $\mathbf{k}^*$ tractable becomes crucial.

and each seller sets $\lambda = 1$, the probability with which a buyer gets served is 0.494. On the other hand, if $m = 6$ and $\lambda = 2$, then $A_B = 0.553$. Finally, with $m = 3$ and $\lambda = 4$, one can verify that $A_B = 0.588$. These results highlight that the intensive margin is more responsive than the extensive one. The intuition is that in a market with less sellers and higher capacity, the congestion effects are more mild.

4.4 Efficiency and Free Entry

So far I have assumed that m is exogenous. With m fixed, the expected surplus in the market depends only on the production announcements of the sellers (see (7)), and I have shown that all equilibria are constrained efficient. If the number of sellers is determined endogenously, by a free entry condition, efficiency is not guaranteed, since a suboptimal number of sellers might enter the market. This is one of the first papers to deal with free entry in a directed search model with finite sellers. The only other paper that tackles this issue, and I am aware of, is Jacquet and Tan (2011). However, their model is quite different, since they study the provision of insurance to risk averse workers.

Suppose that a new stage is added in the beginning of the original game, in which sellers decide whether to enter the market or not. There exists an arbitrarily large number of potential entrants, denoted by $\bar{m}$, and they all get a chance to choose whether they want to enter the market. This happens sequentially and with a random order. Sellers who decide to not enter receive a payoff of zero. Sellers who choose to enter pay an entry fee, $\psi > 0$, in advance, and then they participate in the regular two stage game described earlier. This happens once all potential entrants have announced whether they want to enter the market or not, and the number of sellers who entered is public information.

Let me first describe efficient entry. Suppose that a hypothetical Social Planner chooses the number of firms that will operate in the market and a production plan that has to be followed by each active firm. For any given m, it is trivial to show that the Planner will require firms to post an efficient $\mathbf{k}$. Then, the optimal number of firms operating in the market solves $m^* = \arg\max\{S(m, \mathbf{k}^*) - \psi m\}$, where $S(m, \mathbf{k}^*)$ is described in (7). The term m^* exists, and it is finite.[22] I restrict attention to cases with $m^* \geq 2$. Corollary 2 highlights a set of strategies that lead to equilibrium with optimal entry.

Corollary 2. *Let sellers be indicated by the order in which they decide whether they will enter the market, denoted by $j \in \{1, ..., \bar{m}\}$. Also, let $\pi(m, M)$ denote profit in the symmetric equilibrium where (all) m active sellers post the mechanism M. Then, the following strategies constitute a subgame perfect equilibrium of the game with free entry.*

a) <u>Sellers:</u> Seller j chooses to enter the market if and only if $j \leq m^*$. All sellers post the mechanism

$$M = \begin{cases} M_1 = \{\boldsymbol{p}_1, \boldsymbol{k}^*\}, & \text{if } m = m^*, \\ M_2 = \{\boldsymbol{p}_2, \boldsymbol{k}^*\}, & \text{if } m \neq m^*, \end{cases}$$

[22] The term m^* exists since $m \in \mathbb{N}$. It is finite because for $n < \infty$, $\lim_{m \to \infty} S(m, \mathbf{k}^*) = -\infty$.

where $\mathbf{p}_1, \mathbf{p}_2$ *solve (8) for* $m = m^*$ *and* $m \neq m^*$*, respectively,* $\pi(m^*, M_1) \in [\psi, S(m^*, \boldsymbol{k}^*)/m^*]$*, and* $\pi(m, M_2) \in [0, \psi)$*, for all* $m \neq m^*$*.*

b) *<u>Buyers:</u> All buyers set* $s^* = (1/m, ..., 1/m)$*.*

The strategies identified in Corollary 2 describe equilibrium behavior both on and off the equilibrium path. Thus, the class of equilibria under consideration is subgame perfect. The result utilizes Lemma 1, which states that, regardless of the number of entrants, only efficient $\mathbf{k}$'s will be played in the subgame. Also, it exploits Proposition 4, according to which, for n, m finite, any sharing rule of the surplus can be supported in equilibrium. The equilibrium outcome described in the Corollary is not symmetric, since some sellers enter the market and others do not. However, all sellers play the same strategies.

Althouhg Corollary 2 points out an equilibrium that leads to efficient entry, some equilibria might lead to a suboptimal number of sellers entering the market.[23] For example, assume that $S(m^* + 1, \mathbf{k}^*) - \psi(m^* + 1) > 0$, and consider the following set of strategies: Seller j chooses to enter the market when $j \leq m^* + 1$. All sellers post the mechanism

$$M = \begin{cases} M_1 = \{\mathbf{p}_1, \mathbf{k}^*\}, \text{ if } m = m^* + 1, \\ M_2 = \{\mathbf{p}_2, \mathbf{k}^*\}, \text{ if } m \neq m^* + 1, \end{cases}$$

where $\mathbf{p}_1, \mathbf{p}_2$ solve (8) for $m = m^* + 1$ and $m \neq m^* + 1$, respectively, $\pi(m^* + 1, M_1) \in [\psi, S(m^* + 1, \mathbf{k}^*)/(m^* + 1)]$, and $\pi(m, M_2) \in [0, \psi)$, for all $m \neq m^* + 1$. It can be easily verified that this is a subgame perfect Nash equilibrium which results into $m^* + 1$ sellers entering the market. Clearly, one could construct examples where an inefficiently low number of firms enters. The key behind the existence of all these types of equilibria in the free entry game, is the indeterminacy of the equilibrium sharing rule in the subgame. I Section 5, I show that in infinite markets the indeterminacy of equilibrium vanishes, and the entry of firms is always efficient.

5 The Case of Large Markets

5.1 Equilibrium in Large Markets

In Section 4, it was shown that when n, m are finite, the sharing rule of the equilibrium surplus is not uniquely pinned down. Since this indeterminacy is closely linked to the existence of strategic interaction among sellers (see explanation of Proposition 4), one would expect that it would vanish as the market becomes very large.

Proposition 6. *Suppose* $n, m \to \infty$*, and* $b = n/m$ *is fixed. Let* $e^* < \infty$ *and* $p_i^* > -\infty$*,* $\forall i$*. The limit of the expected profit per seller is unique and given by*

$$\bar{\pi} = b \lim_{n \to \infty} \left\{ \sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right) (1 + b - i)\, \sigma(k_i^*) \right\}. \tag{11}$$

[23] Jacquet and Tan (2011) also find a similar result.

Proof. See the appendix. □

As the market becomes large, considering only fixed price mechanisms, like in BSW, or auctions, like in CE, is without loss of generality. Regardless of what mechanisms are available to sellers, all equilibria yield the same unique $\bar{\pi}$, which is an increasing function of b. The intuition is straightforward. The essence of the indeterminacy result in the finite market, is that the market utility is not constant and sellers have a best response correspondence to a deviant seller who attempts to withdraw customers away from them. However, in a large market, any given seller has only a small impact on the number of customers that other firms receive. So the change on market utility is small and becomes zero in the limit. In this case, the sellers behave as if as they took the rest of the market parametrically (indexed only through the market utility), which is the standard approach in most directed search models and yields a unique equilibrium.

The fact that the sharing rule of the surplus is uniquely pinned down in large markets, does not imply that the equilibrium price schedule is also unique. In the large market case, there is only one equation in order to determine the equilibrium price scheme, which has potentially infinite dimension (since the most general price schedule defines an entry fee and a sequence $\{p_i\}_{i=1}^n$, with $n \to \infty$). This equation comes from the fact that equilibrium profits converge to $\bar{\pi}$, implying that

$$\lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i} [k_i^* \, p_i^* + ie^* - c(k_i^*)] \right\} = \bar{\pi}.$$

Therefore, the equilibrium price schedule will typically not be uniquely pinned down, unless one focuses on fixed price mechanisms (with no entry fee). For example, if $e^* = 0$ and $p_i^* = p^*$ and $k_i^* = 1$ for all i, one can verify that $p^* = (1 - e^{-b} - be^{-b})/(1 - e^{-b})$, which is the result also obtained by BSW.

Returing to Proposition 6, for any given b, $\bar{\pi}$ depends exclusively on the sequence of ex post optimal surplus, $\{\sigma(k_i^*)\}_{i=1}^{\infty}$. With $n \to \infty$, it is not always possible to assign a sequence representation to $\mathbf{k}^*$ and effectively describe $\{\sigma(k_i^*)\}_{i=1}^{\infty}$. This issue can be handled by placing some minor restrictions on the functions u and c, which lead to a very tractable class of efficient production schedules.

Proposition 7. *If $\sigma(x)$ is strictly quasi concave, the unique efficient plan $\boldsymbol{k}^*$ is fully characterized by the number λ, in the sense that $\boldsymbol{k}^* = (1, 2, ..., \lambda - 1, \lambda, \lambda, ...)$. A sufficient condition for $\sigma(x)$ to be strictly quasi concave is that $c(x)$ is convex and $xu(x)$ is concave, one of them strictly.*

Proof. See the appendix. □

If $\sigma(x)$ is strictly quasi concave, $\mathbf{k}^*$ is fully described by a single number λ. If λ or less buyers show up, they all get served, but if more than λ show up, only λ units are sold and rationing takes place. Hence, λ here represents the capacity of the sellers but, unlike most the related literature, here it is endogenously determined. The case in which

$\lambda = \infty$ is not excluded. This is the no frictions case. Henceforth, I restrict attention to preferences and technology that satisfy the assumptions of Proposition 7. The sequence $\{\sigma(k_i^*)\}_{i=1}^{\infty}$ has $\sigma(k_i^*) = \sigma(i)$, for $i \leq \lambda$, and $\sigma(k_i^*) = \sigma(\lambda)$, for $i > \lambda$. Next, I examine some specific examples of $\{\sigma(k_i^*)\}_{i=1}^{\infty}$ and obtain closed form solutions for $\bar{\pi}$.

5.2 Closed Form Solutions

First, consider the case in which $\lambda = 1$, and so $\{\sigma(k_i^*)\}_{i=1}^{\infty}$ is the constant sequence $\{\sigma(1), \sigma(1), ...\}$.[24] One can re-write (11) as

$$\bar{\pi} = b\,\sigma(1) \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right)(1 + b - i) \right\}.$$

It is straightforward to show that

$$\bar{\pi} = \sigma(1)\left(1 - e^{-b} - be^{-b}\right).$$

The expression $1 - e^{-b} - be^{-b}$ is the limiting value of expected profit found in BSW. In that paper, it is assumed that $u(1) = 1$ and $c(1) = 0$. Thus, the two results coincide.

Next, consider the case of a strictly convex cost. For simplicity, let $u(i) = 1$ for all i and $c(i) = \beta\, i^2$, $\beta \in (0, 1)$. Here $\sigma(i) = i(1 - \beta\, i)$. The critical number, above which the marginal cost of serving another customer exceeds the marginal benefit of doing so, is given by $\lambda = \min\{l \in \mathbb{N} : l > (1 - \beta)/(2\beta)\}$. One can show that

$$\begin{aligned} \bar{\pi} &= b \lim_{n\to\infty} \left\{ \sum_{i=1}^{\lambda} \binom{n-1}{i-1} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} (1 + b - i)(1 - \beta\, i) \right\} \\ &+ b\,\lambda(1 - \beta\lambda) \lim_{n\to\infty} \left\{ \sum_{i=\lambda+1}^{n} H\left(n, \frac{b}{n}, i\right)(1 + b - i) \right\}. \end{aligned}$$

After using some standard results from the theory of limits, one can express $\bar{\pi}$ exclusively as a function of λ, β, and b,

$$\bar{\pi} = \lambda(1 - \beta\lambda)\left(1 - e^{-b} - be^{-b}\right) + e^{-b} \sum_{i=1}^{\lambda} \frac{b^i(1 + b - i)}{(i-1)!} \left[1 - \beta\, i - \frac{\lambda(1 - \beta\lambda)}{i}\right].$$

In Figure 2, I plot $\bar{\pi}$ against b for two cases: i) $\beta = 0.3$ (implying $\lambda = 2$), and ii) $\beta = 0.1$ (implying $\lambda = 5$). Notice that as b becomes large, $\bar{\pi} \to \lambda(1 - \beta\lambda)$.

An interesting case arises when $u(i) = u$ for all i, and the cost is linear, $c(i) = c\, i$.

[24] This example coincides with the environment described in BSW. However, in that paper sellers have no choice over their production, while here $\lambda = 1$ because $\sigma(i) < \sigma(1)$, $\forall i$.

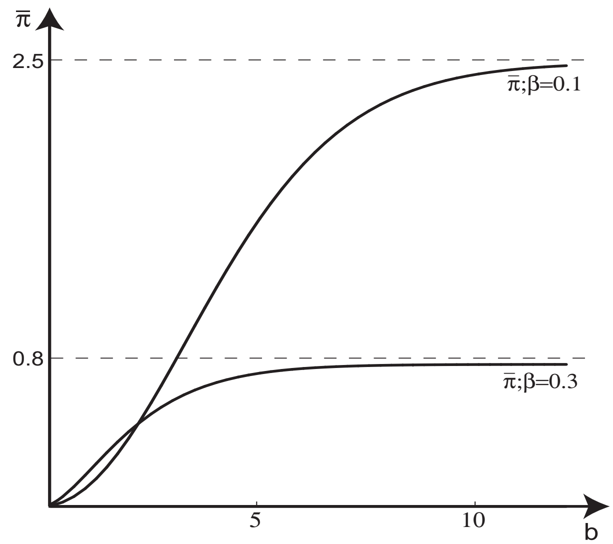

Figure 2: $\bar{\pi}$ as a function of b, for $\beta = 0.3$ and $\beta = 0.1$.

Assuming $c < u$, sellers set $\lambda = \infty$ and $k_i^* = i$. Thus, (11) yields

$$\begin{aligned} \bar{\pi} &= b\,(u-c) \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \binom{n-1}{i-1} \left(1-\frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} (1+b-i) \right\} = \\ &= b\,(u-c)(1+b) \left\{ \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \binom{n-1}{i-1} \left(1-\frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} \right\} - 1 \right\} = 0. \end{aligned}$$

This finding has an intuitive explanation. A crucial feature of every directed search model, is that sellers offer to the buyers different pairs (tradeoffs) of prices and probabilities of service. When $k_i^* = i$, buyers get served with probability 1 at every store, and competition over the probability of service vanishes. All the competition is then transferred on the price end, and sellers are essentially competing *a la Bertrand*, which drives equilibrium profits down to zero.

5.3 Matching in Large Markets

In this subsection, I consider the number of successful matches in a large market. Since the number of buyers and sellers is infinite, I focus on the arrival rates A_S and A_B introduced in Section 4.3.

Proposition 8. *Let λ denote the maximum production of sellers in the symmetric equilibrium. As $n, m \to \infty$, and b is held constant, the expected number of sales per seller is* *given by*

$$\bar{A}_{S,\lambda}(b) \equiv \lim_{n\to\infty} A_S = \lambda(1-e^{-b}) + e^{-b}\sum_{i=1}^{\lambda}\frac{(i-\lambda)\,b^i}{i!},$$

for $\lambda < \infty$, *and* $\bar{A}_{S,\lambda}(b) = b$ *otherwise. The probability with which an arbitrary buyer get served is given by* $\bar{A}_{B,\lambda}(b) \equiv \lim_{n\to\infty} A_B = b^{-1}\bar{A}_{S,\lambda}(b)$.

Proof. See the appendix. □

In large markets, the matching technology exhibits approximately CRS. For any given b, $\lambda > \lambda'$ implies $\bar{A}_{S,\lambda}(b) \geq \bar{A}_{S,\lambda'}(b)$ and $\bar{A}_{B,\lambda}(b) \geq \bar{A}_{B,\lambda'}(b)$.[25] As b becomes large, $\bar{A}_{S,\lambda} \to \lambda$. Figure 3 illustrates $\bar{A}_{S,\lambda}$ and $\bar{A}_{B,\lambda}$ for different values of λ, including $\lambda = \infty$. To see the effect of different choices of λ on the number of successful matches, suppose that $b = 2$. The average expected number of sales in a market where $\lambda = 1$, is $\bar{A}_{S,1}(2) = 0.86$. If $\lambda = 2$, the number of sales is $\bar{A}_{S,2}(2) = 1.46$. From a buyer's perspective, in a market with $\lambda = 1$, the probability of getting served is given by $\bar{A}_{B,1}(2) = 0.43$. But if $\lambda = 2$, this probability increases to $\bar{A}_{B,2}(2) = 0.73$. If b is small, the probability with which an arbitrary buyer gets served in equilibrium is very close to 1, even for values of λ as small as 3. For instance, $\bar{A}_{B,3}(0.6) = 0.99$.

One might claim that, for given b, the distance between $\bar{A}_{B,2}$ and $\bar{A}_{B,1}$ is due to the difference in the total number of goods per buyer in the economy. This is partly true. However, another part of this distance is the result of frictions. This is depicted in Figure 4, where I plot $\bar{A}_B$ as a function of the total quantity of goods per buyer, λ/b, in markets with $\lambda = 1, 2$, and 4. As λ/b increases, $\bar{A}_B$ eventually reaches 1 regardless of λ. However, for any value of λ/b, the probability of getting served is always higher for higher values of λ, which implies that frictions amount for a lot of unsuccessful matches. For $\lambda = 4$ and a value of λ/b as low as 3, $\bar{A}_B = 0.99$. For the same value of λ/b but $\lambda = 2$ or 1, $\bar{A}_B$ goes down to 0.95 or 0.85, respectively. With $\lambda = 1$, one would need $\lambda/b = 50$ in order for $\bar{A}_B$ to be equal to 0.99. These examples highlight that the number of successful matches is more responsive along the intensive margin.

5.4 Free Entry in Large Markets

As in Section 4.4, assume that sellers pay a sunk cost ψ in order to enter the market. What is different compared to the finite market case, is that the equilibrium profit (for a given market tightness) is now uniquely pinned down. Hence, in the symmetric equilibrium, firms will enter the market until the net profit is equal to zero, i.e. until $\bar{\pi}$ given by (11) is equal to ψ. In order to describe the decentralized economy outcome more sharply, let me focus on parameters that satisfy the conditions in Proposition 7. Each firm chooses

[25] Also, $\lim_{\lambda\to\infty} \bar{A}_{S,\lambda}(b) = b$ and $\lim_{\lambda\to\infty} \bar{A}_{B,\lambda}(b) = 1$, which are the arrival rates in a market with no frictions.

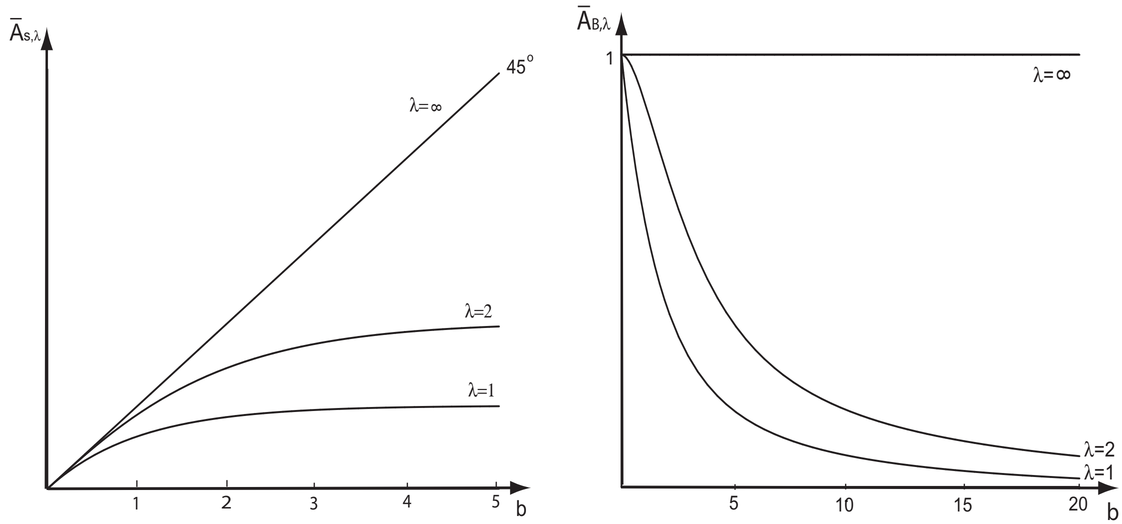

Figure 3: $\bar{A}_{S,\lambda}$ and $\bar{A}_{B,\lambda}$ for $\lambda = 1, 2, \infty$.

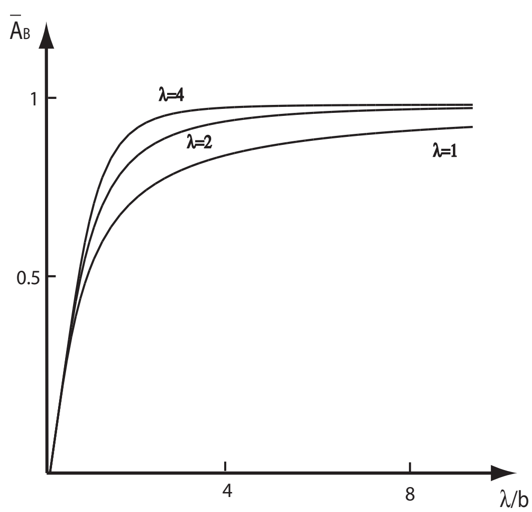

Figure 4: The probability $\bar{A}_B$ for a fixed number of units per buyer.

the ex post efficient production plan described in the proposition ($k_i^* = i$, for $i = 1, .., \lambda$, and $k_i^* = \lambda$ for $i > \lambda$), and the market tightness is uniquely pinned down by the condition

$$\sigma(\lambda)\left(1 - e^{-b} - be^{-b}\right) + e^{-b}\sum_{i=1}^{\lambda}\frac{b^i(1+b-i)}{i!}\left[\sigma(i) - \sigma(\lambda)\right] = \psi. \tag{12}$$

This condition emerges after substituting for the ex post efficient $\mathbf{k}^*$ in (11). The left-hand side, which is the limiting profit, is strictly increasing in b and, hence, the equilibrium market tightness is uniquely determined, as long as $\psi < \sigma(\lambda)$.[26]

I now state the main result of this subsection.

Proposition 9. *Normalize the measure of buyers to one, and let $1/b$ denote the measure of sellers who enter the market. Assume that a benevolent Social Planner chooses the market tightness and a production plan for (all) active sellers, in order to maximize the expected total surplus. Let the Planner's choices be denoted by b^{SP} and $\boldsymbol{k}^{SP}$. Then, $k_i^{SP} = i$, for $i = 1, .., \lambda$ and $k_i^{SP} = \lambda$ for $i > \lambda$, and b^{SP} solves (12).*

Proof. See the appendix. □

Since the proof contains some technical details it is relegated to the appendix. However, the intuition behind this result is standard. In all directed search models search externalities are internalized, since sellers face a tradeoff between the utility they promise to buyers and the probability of getting visited by them. This is not the case in random search models (see for example Mortensen and Pissarides (1994)), where the terms of trade are determined through Nash bargaining. Hence, in directed search models, the so-called Hosios condition for efficiency (Hosios (1990)) is always satisfied. The fact that, in this model, sellers post more general production schemes, complicates the proof but does not alter the basic intuition behind efficiency of entry.

6 Conclusions

In this paper, I extend the standard directed search model in two important directions. First, I allow sellers to post production announcements that specify the number of buyers that get served, for any given number of visiting customers. Second, I allow sellers to advertise a very general class of pricing mechanisms. Hence, given ex post realized demand at a particular location, the seller's advertisement specifies how many customers get served, and the side payments made by buyers who got served and those who did not. I show that equilibrium production announcements are constrained efficient, even in the

[26] This condition guarantees that a positive measure of sellers will enter the market, since the limit of the left-hand side of (12), as b grows large, is $\sigma(\lambda)$. The idea can be seen graphically in Figure 2. In the convex cost example, if $\beta = 0.3$, implying $\lambda = 2$, one would need to impose $\psi < 0.8$ to guarantee entry. On the other hand, if $\beta = 0.1$, implying $\lambda = 5$, any $\psi < 2.5$ would suffice.

case of an oligopolistic market with strategic interaction among sellers. In a variation of the baseline model, I consider free entry and endogenous determination of the number of sellers, and I find that efficiency can be still achieved in equilibrium.

In the case of small markets, indeterminacy of equilibrium arises: a continuum of equilibrium prices can be supported, that are all constrained efficient, but not payoff equivalent. I present an easy method of characterizing the full set of equilibria for any parameter values. Although sellers are allowed to advertise general pricing mechanisms, common practices, like posting a fixed price or an auction, can describe equilibrium behavior. The indeterminacy result has its roots in the strategic interaction among the sellers. I prove that, as the market grows large, the sharing rule of the surplus is uniquely pinned down. Therefore, in the case of large markets, considering more simple pricing mechanisms is without loss of generality.

The paper also examines the number of successful matches in the economy. In the case of small markets, I provide a closed form solution for the matching function for any possible parameter values. I show that, in contrast to findings in related literature, the matching technology need not exhibit DRS. In the case of large markets, a set of minor restrictions on preferences and technology allow me to effectively describe the production announcements of sellers. I find closed form solutions for the arrival rates of buyers and sellers and show that the number of successful matches is more responsive along the intensive margin than it is along the extensive margin.

A Appendix

Proof of Lemma 1. Consider a seller who advertises $M = \{\mathbf{p}, \mathbf{k}\}$ and gets visited by an arbitrary buyer with probability θ. Combine (1) and (2) with (3), and use the fact that $\binom{n}{i} = \binom{n-1}{i-1}(n/i)$. Then, the expected utility of a buyer who visits that seller and the expected profit of the seller can be written as

$$U(\theta, M) = \sum_{i=1}^{n} H(n, \theta, i) k_i \left[u(k_i) - p_i\right] - e. \tag{a.1}$$

$$\pi(\theta, M) = n\theta \Big\{ \sum_{i=1}^{n} H(n, \theta, i) \left[k_i p_i - c(k_i)\right] + e \Big\}. \tag{a.2}$$

Keeping M^{-j} fixed, for any $M \in S^j$, indifference in the second stage determines the probabilities with which an arbitrary buyer visits each seller, $(\theta_1, \theta_2, ..., \theta_{m-1})$. Assume that $\theta_j > 0$ and define $\delta \equiv U_j(\theta_j, M) = U_h(\theta_h, M^{-j}) \geq [0, \infty)$. I claim that, for any $M \in S^j$, one can always find $\mathbf{p}^* = (p_1^*, p_2^*, ..., p_n^*, e^*)$, with $p_i^* \leq u(k_i^*)$ for all i and $e^* \in \mathbb{R}$, such that the equation

$$G(\mathbf{p}^*) \equiv U_j(\theta_j, M^*) - \delta = \sum_{i=1}^{n} H(n, \theta_j, i) k_i^* \left[u(k_i^*) - p_i^*\right] - e^* - \delta = 0, \tag{a.3}$$

has a solution.[27] To see why this is true, fix the entry fee at some arbitrary $e^* = \tilde{e} > 0$. Consider the scheme $\tilde{\mathbf{p}} \equiv (p_1^*, p_2^*, ..., p_n^*, e^*) = (u(k_1^*), u(k_2^*), ..., u(k_n^*), \tilde{e})$. First, $G(\tilde{\mathbf{p}}) = -\tilde{e} - \delta < 0$. Second, one can always find prices that make G arbitrary large. Third, the function G is continuous and decreasing in all arguments. Combining these observations, one can conclude that $G(\mathbf{p}^*) = 0$ has at least one solution.

Let $\hat{\mathbf{p}}^* \in \{\mathbf{p}^* : G(\mathbf{p}^*) = 0\}$. By definition, if seller j advertises the mechanism $\hat{M} = (\hat{\mathbf{p}}^*, \mathbf{k}^*)$, buyers are indifferent between visiting her or any other seller. Given M^{-j}, the profit that seller j obtains if she plays this strategy is

$$\pi_j\left(\theta_j, \hat{M}; M^{-j}\right) = n\theta_j \Big\{ \sum_{i=1}^{n} H(n, \theta_j, i)\left[k_i^* \hat{p}_i^* - c(k_i^*)\right] + e^* \Big\}$$

$$= n\theta_j \Big\{ \sum_{i=1}^{n} H(n, \theta_j, i)\left\{k_i^* u(k_i^*) - c(k_i^*) - k_i\left[u(k_i) - p_i\right]\right\} + e \Big\},$$

where the last equality follows from (a.3) and the definition of δ. Adding and subtracting $c(k_i)$ inside every term of the sum in the last expression implies

$$\begin{aligned}\pi_j\left(\theta_j, \hat{M}; M^{-j}\right) &= n\theta_j \left\{ \sum_{i=1}^{n} H(n, \theta_j, i)\left[\sigma(k_i^*) - \sigma(k_i) + p_i k_i - c(k_i)\right] + e \right\} \\ &= n\theta_j \sum_{i=1}^{n} H(n, \theta_j, i)\left[\sigma(k_i^*) - \sigma(k_i)\right] + \pi_j\left(\theta_j, M; M^{-j}\right),\end{aligned}$$

where the last equality follows from (a.2). Definition 4 and the fact that $\mathbf{k}^*$ is ex post efficient conclude the proof. □

Proof of Proposition 4. a) To characterize equilibrium prices, suppose all sellers but j post $\mathbf{p} = (p_1, ..., p_n, e)$ and the ex post efficient $\mathbf{k}^*$. Seller j also chooses $\mathbf{k}^*$ and considers deviations in prices. If she posts $\mathbf{p}^d = (p_1^d, ..., p_n^d, e^d)$, her expected profit is

$$\pi_j\left(t, (\mathbf{p}^d, \mathbf{k}^*)\right) = nt \Big\{ \sum_{i=1}^{n} H(n, t, i)\left[k_i^* p_i^d - c(k_i^*)\right] + e^d \Big\},$$

where t is the probability with which an arbitrary buyer visits seller j, and H was defined in (3). Seller j wants to maximize this expression, subject to $U\left(t, (\mathbf{p}^d, \mathbf{k}^*)\right) = U\left(\theta, (\mathbf{p}, \mathbf{k}^*)\right)$, where $\theta = (1-t)/(m-1)$ is the probability with which any given buyer visits a non deviant seller. Using (a.1), the condition for indifference of the buyers can be written as

$$\sum_{i=1}^{n} H(n, t, i) k_i^* p_i^d + e^d = \sum_{i=1}^{n} H(n, t, i) k_i^* u(k_i^*) - U\left(\theta, (\mathbf{p}, \mathbf{k}^*)\right).$$

[27] In words, for any choice of M one can find a price schedule $\mathbf{p}^*$ such that the mechanisms $M = \{\mathbf{p}, \mathbf{k}\}$ and $M^* = \{\mathbf{p}^*, \mathbf{k}^*\}$, where $\mathbf{k}^*$ is ex post efficient, generate the same expected utility for a buyer who visits seller j, while leaving θ_j unaltered.

The term $\sum_{i=1}^{n} H(n,t,i)k_i^* p_i^d + e^d$ appears both in the objective function and in the constraint. Hence, one can re-write the problem of seller j as

$$\max_t \left\{ nt \left[\sum_{i=1}^{n} H(n,t,i)\sigma(k_i^*) - U\left(\theta, (\mathbf{p}, \mathbf{k}^*)\right) \right] \right\}.$$

The first-order condition yields

$$t \left\{ \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{\sigma(k_i^*)}{i} (1-t)^{n-i-1}\, t^{i-2} [-(n-i)t + (i-1)(1-t)] - \frac{\partial U}{\partial t} \right\} =$$

$$= U\left(\theta, (\mathbf{p}, \mathbf{k}^*)\right) - \sum_{i=1}^{n} H(n,t,i)\sigma(k_i^*), \tag{a.4}$$

and total differentiation in (a.1) implies

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{k_i^*[u(k_i^*) - p_i][-(n-i)\theta + (i-1)(1-\theta)]}{i(1-\theta)^{1+i-n}\, \theta^{2-i}} \frac{\partial \theta}{\partial t}, \tag{a.5}$$

where $\partial\theta/\partial t = -(m-1)^{-1}$. Combining (a.4) with (a.5), and imposing symmetry, i.e. $t = \theta = 1/m$, $p_i = p_i^d = p_i^*$, $\forall i$, and $e = e^d = e^*$, leads to (8).

b) Without loss of generality, I show existence of $\mathbf{p}^*$ that leads to equilibria with $U(M^*) = 0$. Since e^* enters (8) in a linear fashion, one can always decrease its value and achieve an equilibrium with $U(M^*)$ anywhere between zero and $S(\mathbf{k}^*)$. In this case $\sum_{i=1}^{n} H\left(n, 1/m, i\right) k_i^* \left[u(k_i^*) - p_i^*\right] = e^*$. Using this fact in (8) implies

$$\sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right) \frac{1 - f(n,m,i)}{m-1} k_i^* \left[u(k_i^*) - p_i^*\right] = \sum_{i=1}^{n} Q(n,m,i)\sigma(k_i^*) \equiv \eta, \tag{a.6}$$

where $Q(n,m,i) \equiv H(n, 1/m, i) f(n,m,i)$.

I claim that $\eta \in (0, \infty)$. The definition of H immediately implies $\eta < \infty$, but showing that $\eta > 0$ is not straightforward, because f is not always positive. If $n < m$, then $f(n,m,i) = (mi - n)/(m-1) > 0$ for all i. But if $n > m$, there exists $\nu \in \mathbb{N}$ with $1 \leq \nu < n$, such that $\forall i \leq n$, $f(n,m,i) < 0$ and, therefore, $Q(n,m,i) < 0$. However, one can conveniently re-arrange terms and show that

$$\sum_{i=1}^{n} Q(n,m,i) = \sum_{i=1}^{\nu} Q(n,m,i) + \sum_{i=\nu+1}^{n} Q(n,m,i) = \left(\frac{m-1}{m}\right)^{n-1} > 0. \tag{a.7}$$

Also, since $\mathbf{k}^*$ is ex post efficient, $\sigma(k_i^*)$ is non-decreasing in i. Hence,

$$-\sum_{i=1}^{\nu} Q(n,m,i)\, \sigma(k_i^*) \leq \sigma(k_\nu^*) \sum_{i=1}^{\nu} [-Q(n,m,i)] <$$

$$< \sigma(k_\nu^*) \sum_{i=\nu+1}^{n} Q(n,m,i) \leq \sum_{i=\nu+1}^{n} Q(n,m,i)\, \sigma(k_i^*),$$

where the strict inequality follows from (a.7). This verifies the claim.

Finally, define the left-hand side of (a.6) as $\Gamma(\mathbf{p}^*)$. This function is continuous in all arguments. If $p_i^* = u(k_i^*)$ for all i, then $\Gamma(\mathbf{p}^*) = 0$. Also, I can choose a very large and negative p_i^* in the states for which $[1 - f(n, m, i)]/(m - 1) > 0$, leading to $\Gamma(\mathbf{p}^*) \to \infty$. Combining these facts proves that there always exists a $\mathbf{p}^*$ such that $\Gamma(\mathbf{p}^*) = \eta$. This concludes the proof. A sufficient condition for existence is that $k_i^* \neq 0$ for all i (which is true if $\sigma(1) > 0$). However, even if $k_i^* = 0$ for some i, a symmetric equilibrium can still exist. The problem arises when $k_i^* = 0$ in all states where $[1 - f(n, m, i)]/(m - 1) > 0$. Then $\Gamma(\mathbf{p}^*) \leq 0 < \eta$, and the above proof does not go through.

□

Proof of Proposition 5. Recall that $U(M^*) = 0$ implies

$$\sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right) k_i^* \left[u(k_i^*) - p_i^*\right] = e^*. \tag{a.8}$$

Ex post rationality implies that the left-hand side of (a.8) is non-negative. Since sellers cannot charge $e^* > 0$, $U(M^*) = 0$ can be true only if $e^* = 0$ and $p_i^* = u(k_i^*)$, all i. But $\hat{\mathbf{p}}^* = (u(k_1^*), ..., u(k_n^*), 0)$ cannot be supported in equilibrium. To see this point, plug $\hat{\mathbf{p}}^*$ into (8) to obtain

$$\sum_{i=1}^{n} Q(n, m, i)\, \sigma(k_i^*) = 0,$$

where $Q(n, m, i)$ is defined in the proof of Proposition 4. In that proof I show that $\sum_{i=1}^{n} Q(n, m, i)\, \sigma(k_i^*) > 0$. Hence, I have reached a contradiction.

□

Proof of Lemma 2. If $\mathbf{k}^*$ is ex post efficient, it is straightforward to show that for any $\mathbf{k} \neq \mathbf{k}^*$, $S(\mathbf{k}^*) \geq S(\mathbf{k})$, and so $\mathbf{k}^*$ is also ex ante efficient. To prove the converse, assume that $\mathbf{k}^*$ maximizes the total surplus, i.e. $S(\mathbf{k}^*) \geq S(\mathbf{k})$ for any $\mathbf{k} \neq \mathbf{k}^*$. One can write

$$\sum_{i \in I_{\mathbf{k},\mathbf{k}^*}} \binom{n-1}{i-1} \left(1 - \frac{1}{m}\right)^{n-i} \left(\frac{1}{m}\right)^{i-1} \frac{1}{i} \left[\sigma(k_i^*) - \sigma(k_i)\right] \geq 0,$$

where $I_{\mathbf{k},\mathbf{k}^*} \equiv \{i : k_i \neq k_i^*\}$.

There are two possible scenarios. Either $\sigma(k_i^*) - \sigma(k_i) \geq 0$ for all $i \in I_{\mathbf{k},\mathbf{k}^*}$ or $\sigma(k_i^*) - \sigma(k_i) \geq 0$ for some i and $\sigma(k_i^*) - \sigma(k_i) < 0$ for some others, but the non-negative terms overweight the negative ones. If the former is true, then by definition $\mathbf{k}^*$ is ex post efficient and the proof is complete. Hence, it suffices to show that the latter scenario is excluded. Define $\Psi_{\mathbf{k},\mathbf{k}^*} \equiv \{i \in I_{\mathbf{k},\mathbf{k}^*} : \sigma(k_i^*) < \sigma(k_i)\} \subset I_{\mathbf{k},\mathbf{k}^*}$. Assume, by a way of contradiction, that there exists $\mathbf{k}$, such that $\Psi_{\mathbf{k},\mathbf{k}^*} \neq \emptyset$, and $\mathbf{k}$ is ex post efficient. Consider the plan $\mathbf{k}'$, where

$$k_i' = \begin{cases} k_i, \textit{ if } i \in \Psi_{\mathbf{k},\mathbf{k}^*}, \\ k_i^*, \textit{ otherwise.} \end{cases}$$

By construction, for all $i = 1, 2, ..., n$, $\sigma(k_i') \geq \sigma(k_i^*)$, with strict inequality for some i's (the ones in the non-empty set $\Psi_{\mathbf{k},\mathbf{k}^*}$). This implies that $S(\mathbf{k}') > S(\mathbf{k}^*)$, a contradiction to the fact that $\mathbf{k}^*$ is ex ante efficient. □

Proof of Proposition 6. Suppose that in the symmetric equilibrium all sellers post $M^* = \{\mathbf{p}^*, \mathbf{k}^*\}$, with $\mathbf{k}^*$ ex post efficient. An arbitrary seller's profit is

$$\pi\left(M^*\right) = \frac{n}{m}\Big\{\sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right)\left[k_i^* p_i^* - c(k_i^*)\right] + e^*\Big\}, \tag{a.9}$$

Equilibrium prices in a small market satisfy equation (8). Using this fact, one can re-write (a.9) as

$$\begin{aligned} &\sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right)\left\{k_i^* p_i^* \left[1 - b\,\frac{f(n,m,i)-1}{n-b}\right] - c(k_i^*)\right\} + e^* = \\ &= \sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right)\left\{\left(1 + \frac{b - \frac{ib}{n}}{1-\frac{b}{n}} - i\right)\left[\frac{1}{1-\frac{b}{n}} k_i^* u(k_i^*) - c(k_i^*)\right]\right\}. \end{aligned} \tag{a.10}$$

Define Λ as the left-hand side of equation (a.10). It can be shown that

$$\lim_{n\to\infty} \Lambda = \lim_{n\to\infty}\left\{\sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right)\left[k_i^* p_i^* - c(k_i^*)\right]\right\} + e^*. \tag{a.11}$$

Taking the limit as $n \to \infty$ in equation (a.9) implies that

$$\bar{\pi} = b \lim_{n\to\infty}\left\{\sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right)\left[k_i^* p_i^* - c(k_i^*)\right] + e^*\right\}.$$

The limiting value of profit is equal to the right-hand side of equation (a.11) multiplied by b. Combining this observation with equation (a.10) yields

$$\begin{aligned} \bar{\pi} &= b \lim_{n\to\infty}\left\{\sum_{i=1}^{n} H\left(n, \frac{1}{m}, i\right)\left\{\left(1 + \frac{b - \frac{ib}{n}}{1-\frac{b}{n}} - i\right)\left[\frac{1}{1-\frac{b}{n}} k_i^* u(k_i^*) - c(k_i^*)\right]\right\}\right\} \\ &= b \lim_{n\to\infty}\left\{\sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right)(1 + b - i)\,\sigma(k_i^*)\right\}. \end{aligned}$$

□

Proof of Proposition 7. Suppose $\sigma(1) > 0$ for simplicity. Since the domain of σ is $\mathbb{N}$, quasi concavity implies that for all $h, i, j \in \mathbb{N}$, with $h < j$ and $h \le i \le j$, we have $\sigma(i) > \min\{\sigma(h), \sigma(j)\}$. Suppose, by a way of contradiction, that the ex post efficient schedule $\mathbf{k}^*$ does not have the form described in Proposition 7. The contradictory statement can be re-phrased as follows. Let λ be the smallest number for which $k^*_\lambda = \lambda$ and $k^*_i = \lambda$, where $i = \lambda + 1$.[28] However, there exists $j > i$, such that $k^*_j \neq \lambda$. Since $j > i > \lambda$ and $\mathbf{k}^*$ is ex post efficient, $k^*_j \neq \lambda$ can only imply $k^*_j > \lambda$. Sellers will choose $k^*_j > \lambda$ only if $\sigma(j) > \sigma(\lambda)$. But then, since $j > i > \lambda$ and σ is strictly quasi concave, $\sigma(i) > \min\{\sigma(\lambda), \sigma(j)\} = \sigma(\lambda)$, a contradiction to the fact that $k^*_i = \lambda$. □

Proof of Proposition 8. The expected number of sales per seller is given by

$$\bar{A}_S = \lim_{n\to\infty} \sum_{i=1}^{n} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i} k^*_i.$$

If $\lambda = \infty$, $k^*_i = i$ for all i and the result is immediate. If $\lambda < \infty$,

$$\bar{A}_S = \lim_{n\to\infty} \left\{ \sum_{i=1}^{\lambda} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i} i + \lambda \sum_{i=\lambda+1}^{n} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i} \right\}.$$

Add and subtract $\lambda \sum_{i=1}^{\lambda} \binom{n}{i} (1 - b/n)^{n-i} (b/n)^i$ in the last expression to obtain

$$\bar{A}_S = \lim_{n\to\infty} \Big\{ \Omega_1(n, b, \lambda) + \lambda \Omega_2(n, b) \Big\}, \tag{a.12}$$

where I have defined

$$\begin{aligned} \Omega_1(n, b, \lambda) &\equiv \sum_{i=1}^{\lambda} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i} (i - \lambda), \\ \Omega_2(n, b) &\equiv \sum_{i=1}^{n} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i}. \end{aligned}$$

Using some standard results from the theory of limits, one can show that $\lim_{n\to\infty} \Omega_2(n, b) = 1 - e^{-b}$ and

$$\lim_{n\to\infty} \Omega_1(n, b, \lambda) = \sum_{i=1}^{\lambda} \frac{(i - \lambda) b^i}{i!} \lim_{n\to\infty} \left[\frac{n!}{(n - i)!\, n^i} \left(1 - \frac{b}{n}\right)^{n-i} \right].$$

But since $i \le \lambda < \infty$, $\lim_{n\to\infty}\{n!/[(n - i)!\, n^i]\} = 1$. Also, $\lim_{n\to\infty}(1 - b/n)^{n-i} = e^{-b}$. Therefore,

$$\lim_{n\to\infty} \Omega_1(n, b, \lambda) = \sum_{i=1}^{\lambda} \frac{(i - \lambda) b^i}{i!} e^{-b}.$$

Using these observations in (a.12) delivers the desired result. □

[28] If such a number does not exist, then we have the no frictions case, which is part of the class of production schedules under consideration.

Proof of Proposition 9. Showing that, for a given b, the Planner requires each firm to post an ex post (and ex ante) efficient production plan is trivial. I now focus on the optimal choice of b. The Planner's objective is given by

$$\max_{b\geq 0} \Omega(b) \equiv \max_{b\geq 0} \left\{ \lim_{n\to\infty} \sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right) \sigma(k_i^*) - \frac{\psi}{b} \right\}. \tag{a.13}$$

This expression is just the total surplus in (7) divided by n.[29] Since I am focusing on parameters that lead to a quasi-concave surplus function, I can replace $\mathbf{k}^*$ with $k_i^* = i$, for $i = 1, .., \lambda$, and $k_i^* = \lambda$ for $i > \lambda$. Moreover, I can add and subtract the term $\sigma(\lambda) \sum_{i=1}^{\lambda} H\left(n, \frac{b}{n}, i\right)$ in (a.13) to obtain

$$\Omega(b) = \lim_{n\to\infty} \left\{ \sum_{i=1}^{\lambda} H\left(n, \frac{b}{n}, i\right) [\sigma(i) - \sigma(\lambda)] + \sigma(\lambda) \sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right) \right\} - \frac{\psi}{b}.$$

The following two facts can help find a closed form solution for the expression with limits above (get rid of the terms n). First,

$$\sum_{i=1}^{n} H\left(n, \frac{b}{n}, i\right) = \frac{1 - \left(1 - \frac{b}{n}\right)^n}{b},$$

and $\lim_{n\to\infty}(1 - b/n)^n = e^{-b}$. Second,

$$\lim_{n\to\infty} \sum_{i=1}^{\lambda} H\left(n, \frac{b}{n}, i\right) [\sigma(i) - \sigma(\lambda)] = e^{-b} \sum_{i=1}^{\lambda} \frac{b^{i-1}}{i!} [\sigma(i) - \sigma(\lambda)].$$

Using these facts, one can write the objective function as

$$\Omega(b) = e^{-b} \sum_{i=1}^{\lambda} \frac{b^{i-1}}{i!} [\sigma(i) - \sigma(\lambda)] + \sigma(\lambda) \frac{1 - e^{-b}}{b} - \frac{\psi}{b}.$$

Notice that as $b \to 0$, $\Omega(b) \to -\infty$, and as $b \to \infty$, $\Omega(b) \to 0$. Hence, if an interior maximum exists it will satisfy the first-order condition

$$- e^{-b} \sum_{i=1}^{\lambda} \frac{b^{i-1}}{i!} [\sigma(i) - \sigma(\lambda)] + e^{-b} \sum_{i=1}^{\lambda} \frac{(i-1)b^{i-2}}{i!} [\sigma(i) - \sigma(\lambda)] + \sigma(\lambda) \frac{e^{-b} + be^{-b} - 1}{b^2} + \frac{\psi}{b^2} = 0.$$

After some manipulations one can conclude that an interior maximum solves

$$\sigma(\lambda) \left(1 - e^{-b} - be^{-b}\right) + e^{-b} \sum_{i=1}^{\lambda} \frac{b^i(1 + b - i)}{i!} [\sigma(i) - \sigma(\lambda)] = \psi,$$

[29] I divide by n in order to bound the objective function. This is compatible with the fact that the buyers' population is normalized to 1.

which is identical to (12). Hence, the Planner's choice of b coincides with the market tightness determined in the decentralized equilibrium. A strictly positive measure of firms will enter the market as long as ψ is not too large. If it exists, the optimal $b > 0$ is uniquely defined, since the left-hand side of (12) (i.e. the profit of the typical seller) is increasing in b.

□

References

Acemoglu, D., and R. Shimer (1999a): "Efficient Unemployment Insurance," *Journal of Political Economy*, 107(5), 893–928.

——— (1999b): "Holdups and Efficiency with Search Frictions," *International Economic Review*, 40(4), 827–49.

Burdett, K., S. Shi, and R. Wright (2001): "Pricing and Matching with Frictions," *Journal of Political Economy*, 109(5), 1060–1085.

Burguet, R., and J. Sakovics (1999): "Imperfect Competition in Auction Designs," *International Economic Review*, 40(1), 231–47.

Coles, M. G., and J. Eeckhout (2003): "Indeterminacy and directed search," *Journal of Economic Theory*, 111(2), 265–276.

Epstein, L., and M. Peters (1996): "A Revelation Principle For Competing Mechanisms," Working Papers peters-96-02, University of Toronto, Department of Economics.

Galenianos, M., and P. Kircher (2009): "Directed search with multiple job applications," *Journal of Economic Theory*, 144(2), 445–471.

Godenhielm, M., and K. Kultti (2010): "Directed search with endogenous capacity," Discussion paper, mimeo, University of Helsinki.

Hawkins, W. (2006): "Competitive Search, Efficiency, and Multi-worker Firms," *University of Rochester mimeo.*

Hosios, A. J. (1990): "On the Efficiency of Matching and Related Models of Search and Unemployment," *Review of Economic Studies*, 57(2), 279–98.

Jacquet, N., and S. Tan (2011): "Wage-vacancy contracts and coordination frictions," Discussion paper.

Julien, B., J. Kennes, and I. King (2000): "Bidding for Labor," *Review of Economic Dynamics*, 3(4), 619–649.

Lagos, R. (2000): "An Alternative Approach to Search Frictions," *Journal of Political Economy*, 108(5), 851–873.

Lang, K. (1991): "Persistent wage dispersion and involuntary unemployment," *The Quarterly Journal of Economics*, 106(1), 181–202.

Lester, B. (2010): "Directed search with multi-vacancy firms," *Journal of Economic Theory*.

Martimort, D., and L. Stole (2001): "The Revelation and Delegation Principles in Common Agency Games," Discussion paper.

McAfee, R. P. (1993): "Mechanism Design by Competing Sellers," *Econometrica*, 61(6), 1281–1312.

Moen, E. (1997): "Competitive search equilibrium," *Journal of Political Economy*, 105(2), 385–411.

Montgomery, J. D. (1991): "Equilibrium Wage Dispersion and Interindustry Wage Differentials," *The Quarterly Journal of Economics*, 106(1), 163–79.

Mortensen, D., and C. Pissarides (1994): "Job creation and job destruction in the theory of unemployment," *The review of economic studies*, 61(3), 397.

Peters, M. (1995): "A Competitive Distribution of Auctions," Working Papers peters-95-03, University of Toronto, Department of Economics.

Prescott, E. C. (1975): "Efficiency of the Natural Rate," *Journal of Political Economy*, 83(6), 1229–36.

Shimer, R. (1996): *Essays in search theory*. Massachusetts Institute of Technology, Dept. of Economics.

——— (2007): "Mismatch," *The American Economic Review*, 97(4), 1074–1101.

Tan, S. (2010): "Directed Search and Firm Size," *National University of Singapore mimeo*.

Virag, G. (2007): "Collusive equilibria in directed search models," *University of Rochester mimeo*.

Watanabe, M. (2010): "Middlemen: A Directed Search Equilibrium Approach," *Universidad Carlos III de Madrid mimeo*.